

# FRAMES AND OUTER FRAMES FOR HILBERT $C^*$ -MODULES

LJILJANA ARAMBAŠIĆ AND DAMIR BAKIĆ\*

**ABSTRACT.** The goal of the present paper is to extend the theory of frames for countably generated Hilbert  $C^*$ -modules over arbitrary  $C^*$ -algebras. In investigating the non-unital case we introduce the concept of outer frame as a sequence in the multiplier module  $M(X)$  that has the standard frame property when applied to elements of the ambient module  $X$ . Given a Hilbert  $A$ -module  $X$ , we prove that there is a bijective correspondence of the set of all adjointable surjections from the generalized Hilbert space  $\ell^2(A)$  to  $X$  and the set consisting of all both frames and outer frames for  $X$ . Building on a unified approach to frames and outer frames we then obtain new results on dual frames, frame perturbations, tight approximations of frames and finite extensions of Bessel sequences.

## 1. INTRODUCTION

A Hilbert  $C^*$ -module over a  $C^*$ -algebra  $A$  is a right  $A$ -module  $X$  equipped with an  $A$ -valued inner product  $\langle \cdot, \cdot \rangle : X \times X \rightarrow A$  such that  $X$  is a Banach space with respect to the norm  $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$ . Recall that the inner product on  $X$  has the properties

- (1)  $\langle x, x \rangle \geq 0$ ;
- (2)  $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ ;
- (3)  $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ ;
- (4)  $\langle x, ya \rangle = \langle x, y \rangle a$ ;
- (5)  $\langle x, y \rangle = \langle y, x \rangle^*$ ;

that are satisfied for all  $x, y, z \in X$  and  $a \in A$ .

A Hilbert  $A$ -module  $X$  is said to be full if the closed linear span of the set  $\{\langle x, y \rangle : x, y \in X\}$  is all of  $A$ . We say that  $X$  is countably generated if there exists a sequence  $(x_n)_n$  in  $X$  such that the closed linear span of the set  $\{x_n a : n \in \mathbb{N}, a \in A\}$  is equal to  $X$ . A subclass AFG consists of algebraically finitely generated Hilbert  $A$ -modules, i.e., those  $X$  for which there exists a finite sequence  $(x_n)_{n=1}^N$  such that  $X = \{\sum_{n=1}^N x_n a_n : a_n \in A\}$ . The most important examples of Hilbert  $C^*$ -modules over a  $C^*$ -algebra  $A$  are:

---

*Date:* July 16, 2015.

*2010 Mathematics Subject Classification.* Primary 46L08; Secondary 42C15.

*Key words and phrases.* Hilbert  $C^*$ -modules, frames.

\* Corresponding author.

- $X = \mathbf{A}$  with the inner product  $\langle a, b \rangle = a^*b$ ;
- $X = \mathbf{A}^N = \mathbf{A} \oplus \dots \oplus \mathbf{A}$ ,  $N \in \mathbb{N}$ , ( $N$  copies of  $\mathbf{A}$ ) with the inner product  $\langle (a_1, \dots, a_N), (b_1, \dots, b_N) \rangle = \sum_{n=1}^N a_n^* b_n$ ;
- $X = \ell^2(\mathbf{A})$  - the generalized Hilbert space over  $\mathbf{A}$ . Recall that  $\ell^2(\mathbf{A})$  consists of all sequences  $(a_n)_n$  of elements of  $\mathbf{A}$  such that the series  $\sum_{n=1}^\infty a_n^* a_n$  converges in norm, and the inner product on  $\ell^2(\mathbf{A})$  is defined by  $\langle (a_n)_n, (b_n)_n \rangle = \sum_{n=1}^\infty a_n^* b_n$ . Given  $a \in \mathbf{A}$ , we shall denote by  $a^{(n)} \in \ell^2(\mathbf{A})$  the sequence  $(0, \dots, 0, a, 0, \dots)$  with  $a$  on the  $n$ -th position and zeros elsewhere.

If  $X$  and  $Y$  are Hilbert  $\mathbf{A}$ -modules we denote by  $\mathbb{B}(X, Y)$  the Banach space of all adjointable operators from  $X$  to  $Y$ . Given an adjointable operator  $T$  we denote by  $R(T)$  and  $N(T)$  the range and the null-space of  $T$ , respectively.

For  $x \in X$  and  $y \in Y$  let  $\theta_{y,x} \in \mathbb{B}(X, Y)$  denote the map defined by  $\theta_{y,x}(z) = y\langle x, z \rangle$ ,  $z \in X$ . The linear span of all  $\theta_{y,x}$ 's is denoted by  $\mathbb{F}(X, Y)$ , while its closure is denoted by  $\mathbb{K}(X, Y)$ . These two classes of adjointable operators are referred to as the classes of "finite rank" and generalized compact operators, respectively. In the case  $Y = X$  we simply write  $\mathbb{F}(X)$ ,  $\mathbb{K}(X)$  and  $\mathbb{B}(X)$ .

For basic facts on Hilbert  $C^*$ -modules we refer the reader to [17, 18, 21, 23].

*Definition 1.1.* Let  $X$  be a Hilbert  $C^*$ -module. A (possibly finite) sequence  $(x_n)_n$  in  $X$  is called a *frame* for  $X$  if there exist positive constants  $A$  and  $B$  such that

$$(1) \quad A\langle x, x \rangle \leq \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle \leq B\langle x, x \rangle, \quad \forall x \in X,$$

where the sum in the middle converges in norm. If only the second inequality in (1) is satisfied, we say that  $(x_n)_n$  is a *Bessel sequence*. The constants  $A$  and  $B$  are called *frame bounds*. If  $A = B = 1$ , i.e., if

$$(2) \quad \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle = \langle x, x \rangle, \quad \forall x \in X,$$

the sequence  $(x_n)_n$  is called a *Parseval frame* for  $X$ .

Notice that when we take for the underlying  $C^*$ -algebra of coefficients the field of complex numbers, i.e., when  $X$  is a Hilbert space, (1) becomes

$$A\|x\|^2 \leq \sum_{n=1}^{\infty} |\langle x_n, x \rangle|^2 \leq B\|x\|^2, \quad \forall x \in X,$$

which means that  $(x_n)_n$  is a standard Hilbert space frame.

Frames for Hilbert spaces were introduced by R.J. Duffin and A.C. Schaefer in 1952. In 1980's frames begun to play an important role in wavelet and Gabor analysis. Since then, frames are an important tool in both theoretical and applied mathematics. Frames for Hilbert  $C^*$ -modules were introduced

by M. Frank and D. Larson; the basic modular frame theory is developed in [9, 10, 11]. In particular, it was proved in Example 3.5 in [10] that frames exist in every finitely or countably generated Hilbert  $C^*$ -module. The proof is based on the Kasparov stabilization theorem ([23], Theorem 15.4.6).

In the rest of this introductory section we summarize basic facts concerning modular frames.

First, observe that we do not require in Definition 1.1 that  $X$  is a full Hilbert  $A$ -module. Also, the underlying  $C^*$ -algebra  $A$  may be non-unital. When this is the case we can consider the minimal unitization  $\tilde{A}$  of  $A$  whose elements we write in the form  $a + \lambda e$ ,  $a \in A$ ,  $\lambda \in \mathbb{C}$ , where  $e$  denotes the unit in  $\tilde{A}$ . It is well known that  $X$  can be regarded as a Hilbert  $\tilde{A}$ -module with the same inner product and the action given by  $x(a + \lambda e) = xa + \lambda x$  for all  $x \in X$ ,  $a \in A$ , and  $\lambda \in \mathbb{C}$ . However, one should keep in mind that  $X$  is never full over  $\tilde{A}$ .

Secondly, we point out that we assume the norm-convergence of the series in (1). Since in each  $C^*$ -algebra a convergent series of positive elements necessarily converges unconditionally, we could index our frame by any countable set instead of  $\mathbb{N}$ . Observe also that the unconditional convergence of the series in (1) implies that the family  $\{\langle x, x_n \rangle \langle x_n, x \rangle : n \in \mathbb{N}\}$  is summable in  $A$  for each  $x$  in  $X$ .

Given a frame  $(x_n)_n$  for a Hilbert  $C^*$ -module  $X$ , we define the analysis operator  $U : X \rightarrow \ell^2(A)$  (resp.  $U : X \rightarrow A^N$  if  $(x_n)_{n=1}^N$  is a finite frame) by

$$Ux = (\langle x_n, x \rangle)_n.$$

It is well known that  $U$  is an adjointable map and that the adjoint operator  $U^*$  - that is called the synthesis operator - is given by

$$U^*((a_n)_n) = \sum_{n=1}^{\infty} x_n a_n, \quad \text{resp. } U^*((a_n)_{n=1}^N) = \sum_{n=1}^N x_n a_n.$$

In particular, if  $A$  is unital, we have  $U^*e^{(n)} = x_n$ ,  $n \in \mathbb{N}$ . Here and in the sequel we denote by  $e$  the unit element in a unital  $C^*$ -algebra.

Furthermore, the above defining condition (1) implies that  $U$  is bounded from below; hence,  $R(U)$  is a closed submodule of  $\ell^2(A)$ . Using Corollary 15.3.9 from [23] we now have  $\ell^2(A) = R(U) \oplus N(U^*)$ . Moreover, by [23, Theorem 15.3.8], the range of  $U^*$  is also closed and since  $U$  is, being bounded from below, an injection, we conclude that  $U^*$  is surjective. These properties imply that  $U^*U$  is an invertible operator in  $\mathbb{B}(X)$ .

In particular, note that the analysis operator  $U$  of a Parseval frame  $(x_n)_n$  is an isometry; hence, when this is the case we have  $U^*U = I$ , where  $I$  is the identity operator on  $X$ .

We are now ready to state a result that provides the most fundamental property of modular frames. This was first proved by M. Frank and D. Larson for Hilbert  $C^*$ -modules over unital  $C^*$ -algebras, but it is easily seen that the result extends to the non-unital case too.

**Theorem 1.2.** *Let  $(x_n)_n$  be a frame for a Hilbert  $C^*$ -module  $X$  with the analysis operator  $U$ . Then*

$$(3) \quad x = \sum_{n=1}^{\infty} x_n \langle (U^*U)^{-1} x_n, x \rangle = \sum_{n=1}^{\infty} (U^*U)^{-1} x_n \langle x_n, x \rangle, \quad \forall x \in X.$$

*In particular,  $(x_n)_n$  is a Parseval frame for  $X$  if and only if*

$$(4) \quad x = \sum_{n=1}^{\infty} x_n \langle x_n, x \rangle, \quad \forall x \in X.$$

Given a frame  $(x_n)_n$  for a Hilbert  $C^*$ -module  $X$ , each sequence  $(y_n)_n$  that satisfies the equality

$$x = \sum_{n=1}^{\infty} x_n \langle y_n, x \rangle, \quad \forall x \in X,$$

is called a dual of  $(x_n)_n$ . In general, a frame  $(x_n)_n$  may possess many duals. The first equality in (3) tells us that the sequence  $((U^*U)^{-1} x_n)_n$  is a dual of  $(x_n)_n$ . This sequence is called the canonical dual frame; namely,  $((U^*U)^{-1} x_n)_n$  is indeed a frame for  $X$  since  $(U^*U)^{-1}$  is an adjointable surjection [2, Theorem 2.5].

Observe also that the first equality in (3) shows that each frame  $(x_n)_n$  for  $X$  generates  $X$ . In particular, a Hilbert  $C^*$ -module that admits frames is necessarily countably generated. Analogously, each Hilbert  $C^*$ -module that possesses a finite frame is an AFG Hilbert  $C^*$ -module.

Theorem 1.2 is a natural generalization of the corresponding result for Hilbert spaces. However, further properties of modular frames cannot be derived by simply following the frame theory for Hilbert spaces. To see what the obstacles are, let us first note two basic facts concerning frames in Hilbert spaces.

*Remark 1.3.* Let  $H$  be a Hilbert space.

- (a) Suppose that  $(x_n)_n$  is a sequence in  $H$  such that the series  $\sum_{n=1}^{\infty} c_n x_n$  converges for each sequence  $(c_n)_n \in \ell^2(\mathbb{C})$ . Then  $(x_n)_n$  is a Bessel sequence ([8, Corollary 3.2.4]).
- (b) Each bounded surjection  $T \in \mathbb{B}(\ell^2(\mathbb{C}), H)$  is the synthesis operator of some frame for  $H$  ([8, Theorem 5.5.5]).

Unfortunately, both statements from the preceding remark can fail if the complex field is replaced by a general  $C^*$ -algebra; that is, the above statements are not generally true for Hilbert  $C^*$ -modules. This is demonstrated by two examples that follow. First we need a lemma.

**Lemma 1.4.** *Let  $A$  be a  $C^*$ -algebra and let  $(t_n)_n$  be a sequence in the multiplier algebra  $M(A)$  of  $A$ . The following two statements are equivalent:*

- (a) *The series  $\sum_{n=1}^{\infty} t_n a_n$  is norm-convergent for each  $(a_n)_n \in \ell^2(A)$ .*
- (b) *The sequence  $(\sum_{n=1}^N t_n t_n^*)_N$  is bounded.*

*Proof.* Let us assume (a). Then  $T : \ell^2(A) \rightarrow A$ ,  $T((a_n)_n) = \sum_{n=1}^{\infty} t_n a_n$  is a well defined  $A$ -linear operator, where  $A$  is regarded as a Hilbert  $C^*$ -module over itself. Consider also, for each  $N \in \mathbb{N}$ , the operators  $T_N : \ell^2(A) \rightarrow A$ ,  $T_N((a_n)_n) = \sum_{n=1}^N t_n a_n$ . Observe that  $T_N$ 's are adjointable operators; their adjoints are given by  $T_N^* a = (t_1^* a, \dots, t_N^* a, 0, 0, \dots)$ ,  $a \in A$ . In particular, all  $T_N$  are bounded. Obviously, the sequence  $(T_N)_N$  converges to  $T$  in the strong operator topology. By the uniform boundedness principle there exists a positive constant  $M$  such that  $\|T_N\| \leq M$  for all  $N \in \mathbb{N}$ , and  $\|T\| \leq M$ .

Recall now that for each  $t \in M(A)$  we have

$$\|t\| = \sup\{\|ta\| : a \in A, \|a\| \leq 1\}.$$

Fix  $N \in \mathbb{N}$ . Then for every  $a \in A$  it holds

$$\left\| \sum_{n=1}^N t_n t_n^* a \right\| = \|T(T_N^* a)\| \leq M \left\| \sum_{n=1}^N a^* t_n t_n^* a \right\|^{\frac{1}{2}} \leq M \|a^*\|^{\frac{1}{2}} \left\| \sum_{n=1}^N t_n t_n^* a \right\|^{\frac{1}{2}}.$$

By taking supremum on both sides over all  $a \in A$ ,  $\|a\| \leq 1$ , we get

$$\left\| \sum_{n=1}^N t_n t_n^* \right\| \leq M \left\| \sum_{n=1}^N t_n t_n^* \right\|^{\frac{1}{2}},$$

and hence

$$\left\| \sum_{n=1}^N t_n t_n^* \right\| \leq M^2.$$

Let us now suppose (b), that is, let  $\left\| \sum_{n=1}^N t_n t_n^* \right\| \leq M$  for some  $M > 0$  and all  $N \in \mathbb{N}$ . Take any sequence  $(a_n)_n \in \ell^2(A)$ . For  $\varepsilon > 0$  we can find  $N_0 \in \mathbb{N}$  such that

$$N_2 \geq N_1 \geq N_0 \Rightarrow \left\| \sum_{n=N_1+1}^{N_2} a_n^* a_n \right\| < \varepsilon.$$

From this we conclude, for all  $N_2 \geq N_1 \geq N_0$ ,

$$\begin{aligned} \left\| \sum_{n=1}^{N_2} t_n a_n - \sum_{n=1}^{N_1} t_n a_n \right\|^2 &= \left\| \sum_{n=N_1+1}^{N_2} t_n a_n \right\|^2 \\ &\leq \left\| \sum_{n=N_1+1}^{N_2} t_n t_n^* \right\| \left\| \sum_{n=N_1+1}^{N_2} a_n^* a_n \right\| \\ &< M \varepsilon. \end{aligned}$$

where the first inequality above is obtained by applying the Cauchy-Schwarz inequality in the Hilbert  $C^*$ -module  $M(\mathbf{A})^{N_2-N_1}$ . Thus,  $(\sum_{n=1}^N t_n a_n)_N$  is a Cauchy sequence, and hence convergent.  $\square$

*Example 1.5.* Here we demonstrate an example of a sequence  $(x_n)_n$  in a Hilbert  $C^*$ -module  $X$  over a  $C^*$ -algebra  $\mathbf{A}$  such that the series  $\sum_{n=1}^\infty x_n a_n$  converges in  $X$  for all  $(a_n)_n \in \ell^2(\mathbf{A})$ , but which is not Bessel.

Take an infinite dimensional separable Hilbert space  $H$  with an orthonormal basis  $(\epsilon_n)_n$ . For  $n \in \mathbb{N}$  let  $e_n$  denote the one-dimensional projection onto the subspace  $\text{span}\{\epsilon_n\}$ .

Consider  $X = \mathbb{B}(H)$  as a Hilbert  $C^*$ -module over itself. Obviously,  $\sum_{n=1}^N e_n e_n^*$  is the orthogonal projection onto  $\text{span}\{\epsilon_1, \dots, \epsilon_N\}$ ; thus, the sequence  $(\sum_{n=1}^N e_n e_n^*)_N$  is bounded. By the preceding lemma, the series  $\sum_{n=1}^\infty e_n a_n$  converges for each sequence  $(a_n)_n \in \ell^2(\mathbb{B}(H))$ . However,  $(e_n)_n$  is not a Bessel sequence in  $X$ . Namely, if it were Bessel, that would imply that the series  $\sum_{n=1}^\infty \langle a, e_n \rangle \langle e_n, a \rangle = \sum_{n=1}^\infty a^* e_n a$  converges in norm for each  $a \in X$ . In particular, this norm-limit should coincide with the strong limit of the series  $\sum_{n=1}^\infty a^* e_n a$  which is, obviously, equal to  $a^* a$ . But, this is impossible for each non-compact operator  $a$  on  $H$ . So, if we put  $x_n = e_n$ ,  $n \in \mathbb{N}$ , the sequence  $(x_n)_n$  has the desired properties.

*Remark 1.6.* A well-known result (the Heuser lemma) on square summable sequences of scalars states: if  $(c_n)_n$  is a sequence of complex numbers such that the series  $\langle (c_n)_n, (a_n)_n \rangle = \sum_{n=1}^\infty c_n a_n$  is convergent for each  $(a_n)_n \in \ell^2(\mathbb{C})$ , then  $(c_n)_n \in \ell^2(\mathbb{C})$ . The preceding example shows that an analogous result does not hold in the generalized Hilbert space  $\ell^2(\mathbf{A})$ . Namely, the sequence  $(x_n)_n$  from the preceding example has the property that the series  $\langle (x_n)_n, (a_n)_n \rangle = \sum_{n=1}^\infty x_n a_n$  is convergent for each  $(a_n)_n \in \ell^2(\mathbf{B}(H))$ , but  $(x_n)_n$  does not belong to  $\ell^2(\mathbf{B}(H))$ .

*Example 1.7.* Here we demonstrate an example of an adjointable surjection from  $\ell^2(\mathbf{A})$  to a Hilbert  $\mathbf{A}$ -module  $X$  which is not the synthesis operator of any frame for  $X$ .

Consider an infinite dimensional separable Hilbert space  $H$  such that  $H = \bigoplus_{n=1}^\infty H_n$ , where  $\dim H_n = \infty$  for each  $n \in \mathbb{N}$ . Note that the elements of  $H$  can be identified as sequences  $(\xi_n)_n$  such that  $\xi_n \in H_n$ ,  $n \in \mathbb{N}$ , and  $\sum_{n=1}^\infty \|\xi_n\|^2 < \infty$ . Let  $X = \mathbb{K}(H)$ , where  $\mathbb{K}(H)$  denotes the  $C^*$ -algebra of all compact operators on  $H$ .

Let  $s_n \in \mathbb{B}(H)$  denote the isometry with the final space  $H_n$  for every  $n \in \mathbb{N}$ . Observe that  $s_n^* s_n = e$  ( $e$  stands for the identity operator on  $H$ ), while  $s_n s_n^* = p_n$ , where  $p_n$  denotes the orthogonal projection onto  $H_n$ . Since the sequence  $(\sum_{n=1}^N p_n)_N$  converges to  $e$  in the strong operator topology, a standard argument shows that the sequence  $(\sum_{n=1}^N p_n a)_N$  converges in norm to  $a$  for each compact operator  $a$ . Thus, for each  $a \in \mathbb{K}(H)$  we have  $\sum_{n=1}^\infty p_n a = a$  in the sense of norm-convergence.

Consider now

$$T : \ell^2(\mathbb{K}(H)) \rightarrow \mathbb{K}(H), \quad T((a_n)_n) = \sum_{n=1}^{\infty} s_n a_n.$$

By Lemma 1.4,  $T$  is well defined. Moreover,  $T$  is an adjointable operator; its adjoint  $T^*$  is given by

$$T^*a = (\langle s_n, a \rangle)_n = (s_n^*a)_n.$$

Note that  $T^*$  is well defined since we have, by the conclusion from the preceding paragraph,

$$\sum_{n=1}^{\infty} a^* s_n s_n^* a = a^* \sum_{n=1}^{\infty} p_n a = a^* a, \quad \forall a \in \mathbb{K}(H).$$

This also shows that  $T^*$  is an isometry; hence,  $T$  is a surjection.

We now claim that there does not exist a frame  $(x_n)_n$  for  $X = \mathbb{K}(H)$  whose synthesis operator is  $T$ . To see this, suppose the opposite: let  $(x_n)_n$  be a frame for  $X$  such that  $T((a_n)_n) = \sum_{n=1}^{\infty} x_n a_n$  for each  $(a_n)_n \in \ell^2(\mathbb{K}(H))$ . Then we have  $\sum_{n=1}^{\infty} x_n a_n = \sum_{n=1}^{\infty} s_n a_n$  for each  $(a_n)_n \in \ell^2(\mathbb{K}(H))$ . In particular, if we take arbitrary  $n \in \mathbb{N}$ ,  $a \in \mathbb{K}(H)$ , and  $a^{(n)} \in \ell^2(\mathbb{K}(H))$ , we get  $x_n a = s_n a$  for all  $n \in \mathbb{N}$  and  $a \in \mathbb{K}(H)$ . Since  $\mathbb{K}(H)$  acts non-degenerately on  $H$ , this is enough to conclude  $x_n = s_n$  for all  $n \in \mathbb{N}$ . But this is obviously impossible since each  $s_n$  is a non-compact operator.

As the above two examples show, both statements from Remark 1.3 can fail in Hilbert  $C^*$ -modules. We shall address these problems more thoroughly at the beginning of Section 3 and in Section 5 in the discussion following Remark 5.4. We will show that in the study of Hilbert  $C^*$ -modules over non-unital  $C^*$ -algebras some difficulties arise from sequences and operators with properties as in the preceding two examples and that these difficulties cannot be circumvented by simply adjoining the unit element to the underlying  $C^*$ -algebra  $A$  and regarding the original Hilbert  $C^*$ -module  $X$  as a module over the unital  $C^*$ -algebra  $\tilde{A}$ .

The paper is organized as follows. In Section 2 we discuss further basic properties of frames. In particular, we describe in Proposition 2.3 and Theorem 2.8 the interrelation of Parseval frames for a Hilbert  $C^*$ -module  $X$  with increasing approximate units for  $\mathbb{K}(X)$ .

In Section 3 we introduce the concept of an outer frame for a Hilbert  $C^*$ -module  $X$ , a concept that naturally fits into the picture when one studies frames for Hilbert  $C^*$ -modules over non-unital  $C^*$ -algebras. We show in Theorem 3.19 that there is a bijective correspondence of the set of all adjointable surjections from the generalized Hilbert space  $\ell^2(A)$  to a Hilbert  $A$ -module  $X$  and the set consisting of all both frames and outer frames for  $X$ .

In Section 4 we describe all frames that are dual to a given frame. It turns out that in these considerations one has to take into account outer

frames discussed in the preceding section. In particular, we describe in Theorems 4.6 and 4.14 (synthesis operators of) all frames and outer frames that are dual to a given frame or an outer frame. At the end of Section 4 we discuss frames and outer frames with a unique dual.

Section 5 is devoted to frame perturbations and tight approximations of frames. Again, outer frames naturally fit into the picture when discussing the non-unital case. After proving a perturbation result (Theorem 5.2), we obtain in Propositions 5.8 and 5.9 the best Parseval resp. tight approximation of a frame or an outer frame in terms of the distance of the corresponding analysis/synthesis operators.

Finally, in the concluding Section 6 we investigate finite extensions of Bessel sequences to frames and outer frames. In Theorems 6.1 and 6.7 we characterize those Bessel sequences that admit such extensions to frames.

Throughout the paper  $A$  will denote an arbitrary  $C^*$ -algebra. We do not assume that  $A$  is unital and this particular assumption will be explicitly stated when needed. The multiplier algebra of  $A$  will be denoted by  $M(A)$ . By an approximate unit for a  $C^*$ -algebra  $A$  we understand a net  $(e_\lambda)_\lambda$  of positive elements in the unit ball of  $A$  such that  $\lim_\lambda e_\lambda a = a$ , for all  $a \in A$ . Approximate unit is increasing if  $e_\lambda \leq e_\mu$  whenever  $\lambda \leq \mu$ . Recall that a  $C^*$ -algebra  $A$  has a countable approximate unit precisely when it is  $\sigma$ -unital (i.e., when there exists a strictly positive element in  $A$ ).

Given a  $C^*$ -algebra  $A$  and the generalized Hilbert space  $\ell^2(A)$  over  $A$ , we denote by  $c_{00}(A)$  the set of all finite sequences in  $\ell^2(A)$ , i.e.

$$c_{00}(A) = \{(a_n)_n : a_n \in A, a_n = 0, \forall n > N \text{ for some } N \in \mathbb{N}\}.$$

Clearly,  $c_{00}(A)$  is norm-dense in  $\ell^2(A)$ .

We tacitly assume that the class of countably generated Hilbert  $C^*$ -modules includes all AFG Hilbert  $C^*$ -modules (and, obviously, when we work with finite frames for AFG modules the convergence questions become superfluous). We shall explicitly indicate when a particular discussion is concerned with AFG modules exclusively.

## 2. BASIC PROPERTIES AND CHARACTERIZATIONS

The frame condition (1) from Definition 1.1 involves two inequalities concerning order in the underlying  $C^*$ -algebra that are not always easy to verify. However, it turns out that it suffices to check the corresponding inequalities in norm ([2, Theorem 2.6] and [15, Proposition 3.8]). In fact, as we shall see in our Theorem 2.2 below, even more is true. Our first theorem is concerned with Bessel sequences. We show that, in order to prove that a sequence  $(x_n)_n$  in a Hilbert  $A$ -module  $X$  is Bessel, one has only to verify that the sequence  $(\langle x_n, x \rangle)_n$  belongs to  $\ell^2(A)$  for each  $x \in X$ .



**Theorem 2.1.** *Let  $(x_n)_n$  be a sequence in a Hilbert  $A$ -module  $X$ . Then the following two conditions are equivalent:*

- (a)  $(x_n)_n$  is a Bessel sequence.
- (b) The series  $\sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle$  converges for all  $x$  in  $X$ .

If  $(x_n)_n$  is a Bessel sequence, its analysis operator

$$U : X \rightarrow \ell^2(A), \quad U(x) = (\langle x_n, x \rangle)_n,$$

is well defined and adjointable and the adjoint operator  $U^*$  is given by

$$(5) \quad U^*((a_n)_n) = \sum_{n=1}^{\infty} x_n a_n, \quad \forall (a_n)_n \in \ell^2(A),$$

where the series  $\sum_{n=1}^{\infty} x_n a_n$  converges unconditionally for all  $(a_n)_n \in \ell^2(A)$ . In particular, if  $(e_\lambda)_\lambda$  is an approximate unit for  $A$ , then  $U^*e_\lambda^{(n)} = x_n e_\lambda$  and  $\lim_\lambda U^*e_\lambda^{(n)} = x_n$  for each  $n \in \mathbb{N}$ . Consequently, the sequence  $(x_n)_n$  is bounded and  $\|x_n\| \leq \|U\|$  for all  $n$  in  $\mathbb{N}$ . Finally, if  $A$  is unital then  $x_n = U^*e^{(n)}$  for all  $n \in \mathbb{N}$ .

*Proof.* Suppose that (b) is satisfied. Then the operator  $U : X \rightarrow \ell^2(A)$ ,  $Ux = (\langle x_n, x \rangle)_n$ , is well defined and  $A$ -linear. We now show that  $U$  has a closed graph. Let  $(y, (a_n)_n) = \lim_{k \rightarrow \infty} (y_k, U y_k)$ , where  $y_k, y \in X$ ,  $(a_n)_n \in \ell^2(A)$ . For each  $m \in \mathbb{N}$  and all  $k \in \mathbb{N}$  we have

$$\begin{aligned} (a_m - \langle x_m, y_k \rangle)^* (a_m - \langle x_m, y_k \rangle) &\leq \sum_{n=1}^{\infty} (a_n - \langle x_n, y_k \rangle)^* (a_n - \langle x_n, y_k \rangle) \\ &= \|(a_n)_n - U y_k\|^2. \end{aligned}$$

Taking norms on both sides we get

$$\|a_m - \langle x_m, y_k \rangle\|^2 \leq \|(a_n)_n - U y_k\|^2.$$

By assumption  $(a_n)_n = \lim_{k \rightarrow \infty} U y_k$  and  $y = \lim_{k \rightarrow \infty} y_k$ , so we get

$$a_m = \lim_{k \rightarrow \infty} \langle x_m, y_k \rangle = \langle x_m, y \rangle.$$

As  $m$  was arbitrary, this shows that  $(a_n)_n = U y$ . So, the graph of  $U$  is closed and hence  $U$  is a bounded operator. Now, by [19, Theorem 2.8], it follows that  $\langle Ux, Ux \rangle \leq \|U\|^2 \langle x, x \rangle$  for all  $x \in X$ ; thus,  $(x_n)_n$  is a Bessel sequence.

Let us now show, for each  $(a_n)_n \in \ell^2(A)$ , the unconditional convergence of the series  $\sum_{n=1}^{\infty} x_n a_n$ . Take arbitrary finite set  $F \subseteq \mathbb{N}$  and denote by  $|F|$

the cardinality of  $F$ . Then

$$\begin{aligned}
\left\| \sum_{n \in F} x_n a_n \right\|^2 &= \sup \left\{ \left\| \left\langle \sum_{n \in F} x_n a_n, y \right\rangle \right\|^2 : y \in X, \|y\| \leq 1 \right\} \\
&= \sup \left\{ \left\| \sum_{n \in F} a_n^* \langle x_n, y \rangle \right\|^2 : y \in X, \|y\| \leq 1 \right\} \\
&\quad (\text{by applying the Cauchy-Schwarz inequality in } \mathbf{A}^{|F|}) \\
&\leq \sup \left\{ \left\| \sum_{n \in F} a_n^* a_n \right\| \left\| \sum_{n \in F} \langle y, x_n \rangle \langle x_n, y \rangle \right\| : y \in X, \|y\| \leq 1 \right\} \\
&\leq \sup \left\{ \left\| \sum_{n \in F} a_n^* a_n \right\| \left\| \sum_{n=1}^{\infty} \langle y, x_n \rangle \langle x_n, y \rangle \right\| : y \in X, \|y\| \leq 1 \right\} \\
&= \left\| \sum_{n \in F} a_n^* a_n \right\| \sup \{ \|Uy\|^2 : y \in X, \|y\| \leq 1 \} \\
&= \|U\|^2 \left\| \sum_{n \in F} a_n^* a_n \right\|.
\end{aligned}$$

Since the series  $\sum_{n=1}^{\infty} a_n^* a_n$  converges unconditionally, the family  $\{a_n^* a_n : n \in \mathbb{N}\}$  is summable. Hence, the inequality  $\left\| \sum_{n \in F} x_n a_n \right\|^2 \leq \|U\|^2 \left\| \sum_{n \in F} a_n^* a_n \right\|$  that we have obtained for each finite subset  $F$  of  $\mathbb{N}$ , shows summability of the family  $\{x_n a_n : n \in \mathbb{N}\}$ , which is equivalent to the unconditional convergence of the series  $\sum_{n=1}^{\infty} x_n a_n$ .

It is now easy to prove that  $U$  is adjointable, since we now know that the operator given in (5) is well defined, and satisfies

$$\langle Ux, (a_n)_n \rangle = \sum_{n=1}^{\infty} \langle x, x_n \rangle a_n = \langle x, \sum_{n=1}^{\infty} x_n a_n \rangle = \langle x, U^*((a_n)_n) \rangle$$

for all  $x \in X$  and  $(a_n)_n \in \ell^2(\mathbf{A})$ .

The remaining assertions are evident.  $\square$

A direct consequence of the preceding theorem is the following characterization of frames for Hilbert  $C^*$ -modules.

**Theorem 2.2.** *Let  $(x_n)_n$  be a sequence in a Hilbert  $\mathbf{A}$ -module  $X$ . Then the following two conditions are equivalent:*

- (a)  $(x_n)_n$  is a frame for  $X$ .
- (b) The series  $\sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle$  converges for all  $x \in X$  and there exists a constant  $A > 0$  such that  $A\|x\|^2 \leq \left\| \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle \right\|$  for all  $x$  in  $X$ .

*Proof.* Immediate from Theorem 2.1 and [2, Theorem 2.6] (or [15, Proposition 3.8]).  $\square$

Another useful characterization of frames arises from a correspondence of Parseval frames for a Hilbert  $C^*$ -module  $X$  with approximate units for  $\mathbb{K}(X)$ .

It is easy to check that, regarding a  $C^*$ -algebra  $A$  as a Hilbert  $C^*$ -module over itself, a sequence  $(a_n)_n$  of elements of  $A$  is a Parseval frame for  $A$  precisely when the sequence  $(\sum_{n=1}^N a_n a_n^*)_N$  is an approximate unit for  $A$ . This observation was extended in [16, Theorem 1.4] to a wider class of Hilbert  $C^*$ -modules. We show in the following proposition that it remains true for all Hilbert  $C^*$ -modules.

**Proposition 2.3.** *Let  $X$  be a Hilbert  $C^*$ -module. Then a sequence  $(x_n)_n$  of elements of  $X$  is a Parseval frame for  $X$  if and only if the sequence  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  is an approximate unit for  $\mathbb{K}(X)$ .*

*Proof.* Suppose that  $(x_n)_n$  is a Parseval frame for  $X$ . Let  $F_N = \sum_{n=1}^N \theta_{x_n, x_n}$ ,  $N \in \mathbb{N}$ . Obviously,  $0 \leq F_N \leq F_{N+1}$  for all  $N \in \mathbb{N}$ . From

$$\langle F_N x, x \rangle = \sum_{n=1}^N \langle x, x_n \rangle \langle x_n, x \rangle \leq \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle = \langle x, x \rangle, \quad \forall x \in X,$$

it follows  $F_N \leq I$  for all  $N \in \mathbb{N}$  (where  $I$  denotes the identity operator on  $X$ ).

Let us now fix  $v$  and  $w$  from  $X$ . For any  $x \in X$  we have

$$\begin{aligned} \|(\theta_{v,w} - F_N \theta_{v,w})x\| &= \left\| v \langle w, x \rangle - \sum_{n=1}^N x_n \langle x_n, v \rangle \langle w, x \rangle \right\| \\ &= \left\| \left( v - \sum_{n=1}^N x_n \langle x_n, v \rangle \right) \langle w, x \rangle \right\|. \end{aligned}$$

Since  $(x_n)_n$  is a Parseval frame for  $X$ , this shows, by the reconstruction property (4) from Theorem 1.2, that  $\|\theta_{v,w} - F_N \theta_{v,w}\| \rightarrow 0$  as  $N \rightarrow \infty$ . Clearly, this implies  $\|\theta - F_N \theta\| \rightarrow 0$  as  $N \rightarrow \infty$  for each  $\theta \in \mathbb{F}(X)$ . Finally, take arbitrary  $T \in \mathbb{K}(X)$  and  $\varepsilon > 0$ . Then we can find  $\theta \in \mathbb{F}(X)$  such that  $\|T - \theta\| < \frac{\varepsilon}{3}$ . Further, there exists  $N_0$  such that  $\|\theta - F_N \theta\| < \frac{\varepsilon}{3}$  whenever  $N \geq N_0$ . Then, for each  $N \geq N_0$  we have

$$\|T - F_N T\| \leq \|T - \theta\| + \|\theta - F_N \theta\| + \|F_N \theta - F_N T\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

so  $(F_N)_N$  is an approximate unit for  $\mathbb{K}(X)$ .

To prove the converse, suppose that  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  is an approximate unit for  $\mathbb{K}(X)$ . Then  $\theta_{y,y} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \theta_{x_n, x_n} \theta_{y,y}$  for each  $y \in X$ . In particular,

$$(6) \quad \theta_{y,y}(y) = \lim_{N \rightarrow \infty} \sum_{n=1}^N \theta_{x_n, x_n} \theta_{y,y}(y), \quad \forall y \in X.$$

Recall from Proposition 2.31 in [21] that each  $x \in X$  can be written in the form  $x = y\langle y, y \rangle = \theta_{y,y}(y)$  for some  $y \in X$ . Then (6) becomes

$$x = \lim_{N \rightarrow \infty} \sum_{n=1}^N x_n \langle x_n, x \rangle, \quad \forall x \in X,$$

so by Theorem 1.2,  $(x_n)_n$  is a Parseval frame for  $X$ .  $\square$

The preceding proposition extends to arbitrary frames in a standard way (see also [16, Theorem 1.4]). In the corollary that follows we shall use the strict convergence in  $\mathbb{B}(X)$  with respect to the ideal of generalized compact operators  $\mathbb{K}(X)$ .

**Corollary 2.4.** *Let  $X$  be a Hilbert  $C^*$ -module. Then a sequence  $(x_n)_n$  of elements of  $X$  is a frame for  $X$  if and only if the sequence  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  strictly converges to some invertible operator in  $\mathbb{B}(X)$ .*

*Proof.* Let  $(x_n)_n$  be a frame for  $X$  and  $U$  its analysis operator. We apply Proposition 2.3 to the Parseval frame  $((U^*U)^{-\frac{1}{2}}x_n)_n$  (the sequence  $((U^*U)^{-\frac{1}{2}}x_n)_n$  is indeed a Parseval frame for  $X$ , see [9]). By Proposition 2.3 we conclude that the sequence  $(\sum_{n=1}^N \theta_{(U^*U)^{-\frac{1}{2}}x_n, (U^*U)^{-\frac{1}{2}}x_n})_N$  is an approximate unit for  $\mathbb{K}(X)$ . From this we deduce that  $(\sum_{n=1}^N \theta_{(U^*U)^{-\frac{1}{2}}x_n, (U^*U)^{-\frac{1}{2}}x_n})_N$  strictly converges to the identity operator  $I$ . Since we have

$$\sum_{n=1}^N \theta_{(U^*U)^{-\frac{1}{2}}x_n, (U^*U)^{-\frac{1}{2}}x_n} = (U^*U)^{-\frac{1}{2}} \left( \sum_{n=1}^N \theta_{x_n, x_n} \right) (U^*U)^{-\frac{1}{2}}, \quad \forall N \in \mathbb{N},$$

it follows that the sequence  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  strictly converges to the invertible operator  $(U^*U)^{-1} \in \mathbb{B}(X)$ .

Conversely, if  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  strictly converges to some invertible  $T \in \mathbb{B}(X)$ , then  $T$  is necessarily positive, so it follows that the increasing sequence  $(\sum_{n=1}^N \theta_{T^{-\frac{1}{2}}x_n, T^{-\frac{1}{2}}x_n})_N$  strictly converges to the identity operator on  $X$ . In other words, the sequence  $(\sum_{n=1}^N \theta_{T^{-\frac{1}{2}}x_n, T^{-\frac{1}{2}}x_n})_N$  is an approximate unit for  $\mathbb{K}(X)$ . By Proposition 2.3 it follows that  $(T^{-\frac{1}{2}}x_n)_n$  is a Parseval frame for  $X$ . Finally, applying [2, Theorem 2.5], we conclude that  $(x_n)_n$  is a frame for  $X$ .  $\square$

Next we show that every countably generated Hilbert  $C^*$ -module  $X$  admits approximate units for  $\mathbb{K}(X)$  of the form as in Proposition 2.3. In the proof we shall make use of the left Hilbert  $C^*$ -module structure on  $X$  arising from the action of generalized compact operators.

**Theorem 2.5.** *Let  $X$  be a countably generated Hilbert  $A$ -module. There exists a sequence  $(x_n)_n$  in  $X$  such that  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  is an approximate unit for  $\mathbb{K}(X)$ .*

*Proof.* Since  $X$  is countably generated over  $A$ , Proposition 6.7 from [17] implies that the  $C^*$ -algebra  $\mathbb{K}(X)$  is  $\sigma$ -unital.

Now recall that  $X$  is also a full left Hilbert  $\mathbb{K}(X)$ -module with the action  $(T, x) \mapsto Tx$ ,  $T \in \mathbb{K}(X)$ ,  $x \in X$ , and the inner product  $[x, y] = \theta_{x, y}$ . The resulting norm  $\mathbb{K}\|x\| = \|\theta_{x, x}\|^{\frac{1}{2}}$  coincides with the original norm on  $X$  that arises from the right module structure over  $A$ .

We now apply Lemma 7.3 from [17] to the full left Hilbert  $\mathbb{K}(X)$ -module  $X$ : there exists a sequence  $(x_n)_n$  in  $X$  such that the sequence  $(\sum_{n=1}^N [x_n, x_n])_N$ , that is,  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$ , is an approximate unit for  $\mathbb{K}(X)$ .  $\square$

Observe that the existence of frames in countably generated Hilbert  $C^*$ -modules can now be reproved by using Theorem 2.5 and Proposition 2.3.

We also have the following easy consequence of Theorem 2.5.

**Corollary 2.6.** *Let  $X$  be a countably generated Hilbert  $A$ -module. For each positive operator  $T \in \mathbb{K}(X)$  there exists a sequence  $(y_n)_n$  in  $X$  such that  $T = \sum_{n=1}^{\infty} \theta_{y_n, y_n}$ , where this series converges in norm.*

*Proof.* Let  $T \in \mathbb{K}(X)$ ,  $T \geq 0$ . Using the approximate unit from the preceding theorem we have  $T^{\frac{1}{2}} = \lim_{N \rightarrow \infty} \sum_{n=1}^N T^{\frac{1}{2}} \theta_{x_n, x_n}$ . Multiplying by  $T^{\frac{1}{2}}$  from the right hand side we get  $T = \lim_{N \rightarrow \infty} \sum_{n=1}^N T^{\frac{1}{2}} \theta_{x_n, x_n} T^{\frac{1}{2}}$ . This shows that  $(y_n)_n$ , where  $y_n = T^{\frac{1}{2}} x_n$ ,  $n \in \mathbb{N}$ , is a sequence with the desired property.  $\square$

The following result shows that approximate units as in Theorem 2.5, although of a very special form, not only exist (provided that  $X$  is countably generated), but can be derived from any increasing countable approximate unit in  $\mathbb{K}(X)$ . To prove this, we first need an auxiliary result on approximate units in  $C^*$ -algebras.

**Lemma 2.7.** *Let  $(e_n)_n$  be a sequence in a  $C^*$ -algebra  $A$  such that  $0 \leq e_n \leq e_{n+1}$  and  $\|e_n\| \leq 1$  for all  $n \in \mathbb{N}$ . If there exists a subsequence  $(e_{p(n)})_n$  of  $(e_n)_n$  which is an approximate unit for  $A$ , then  $(e_n)_n$  is also an approximate unit for  $A$ .*

*Proof.* Let  $a \in A$ . First observe that  $\lim_{n \rightarrow \infty} \|e_{p(n)} a - a\| = 0$  implies  $\lim_{n \rightarrow \infty} \|a^* e_{p(n)} a - a^* a\| = 0$ . Fix  $\varepsilon > 0$  and find  $n_0 \in \mathbb{N}$  such that  $\|a^* e_{p(n)} a - a^* a\| < \varepsilon$  for all  $n \geq n_0$ . Since  $(e_n)_n$  increases and  $\|e_n\| \leq 1$  for all  $n$ , we have

$$0 \leq a^* a - a^* e_n a \leq a^* a - a^* e_{p(n_0)} a, \quad \forall n \geq p(n_0),$$

so

$$(7) \quad \|a^* a - a^* e_n a\| \leq \|a^* a - a^* e_{p(n_0)} a\| < \varepsilon, \quad \forall n \geq p(n_0).$$

We now continue our computation in  $\tilde{A}$ , if needed. Observe that  $\|e - e_n\| \leq 1$  for all  $n \in \mathbb{N}$ . Since

$$\|a - e_n a\|^2 = \|(e - e_n)^{\frac{1}{2}} (e - e_n)^{\frac{1}{2}} a\|^2 \leq \|(e - e_n)^{\frac{1}{2}} a\|^2 = \|a^* a - a^* e_n a\|,$$

(7) gives us  $\lim_{n \rightarrow \infty} \|a - e_n a\| = 0$ .  $\square$

**Theorem 2.8.** *Let  $X$  be a Hilbert  $A$ -module.*

- (a) *If a sequence  $(E_N)_N$  is an increasing approximate unit for  $\mathbb{K}(X)$  then there exists a sequence  $(x_n)_n$  in  $X$  and an increasing sequence of natural numbers  $(p(N))_N$  with the properties  $\sum_{n=1}^{p(N)} \theta_{x_n, x_n} \leq E_N$  and  $\left\| \sum_{n=1}^{p(N)} \theta_{x_n, x_n} - E_N \right\| < \frac{1}{N}$  for all  $N \in \mathbb{N}$ .*
- (b) *If  $(x_n)_n$  is any sequence in  $X$  as in (a), then  $(x_n)_n$  is a Parseval frame for  $X$ .*

*Proof.* To prove (a), suppose that  $(E_N)_N$  is an increasing approximate unit for  $\mathbb{K}(X)$ . By Corollary 2.6, there exists a sequence  $(y_n^1)_n$  in  $X$  such that  $E_1 = \sum_{n=1}^{\infty} \theta_{y_n^1, y_n^1}$ . Find  $p(1)$  such that

$$\left\| E_1 - \sum_{n=1}^{p(1)} \theta_{y_n^1, y_n^1} \right\| < 1.$$

Put  $x_n = y_n^1$  for  $n = 1, 2, \dots, p(1)$  and  $F_{p(1)} = \sum_{n=1}^{p(1)} \theta_{x_n, x_n}$ . Then  $F_{p(1)} \leq E_1$  and  $\|F_{p(1)} - E_1\| < 1$ .

Now observe that  $F_{p(1)} \leq E_1 \leq E_2$  implies  $E_2 - F_{p(1)} \geq 0$ . Again by Corollary 2.6, there exists a sequence  $(y_n^2)_n$  in  $X$  such that  $E_2 - F_{p(1)} = \sum_{n=1}^{\infty} \theta_{y_n^2, y_n^2}$ . Choose  $M$  such that

$$\left\| E_2 - F_{p(1)} - \sum_{n=1}^M \theta_{y_n^2, y_n^2} \right\| < \frac{1}{2}.$$

Denote  $p(2) = p(1) + M$  and  $x_{p(1)+n} = y_n^2$ ,  $n = 1, 2, \dots, M$ . Let  $F_{p(2)} = F_{p(1)} + \sum_{n=1}^M \theta_{y_n^2, y_n^2} = \sum_{n=1}^{p(2)} \theta_{x_n, x_n}$ . Then, by construction, we have  $F_{p(2)} \leq E_2$  and  $\|F_{p(2)} - E_2\| < \frac{1}{2}$ .

Proceed by induction to obtain  $F_{p(N)} = \sum_{n=1}^{p(N)} \theta_{x_n, x_n}$  with the properties  $F_{p(N)} \leq E_N$  and  $\|F_{p(N)} - E_N\| < \frac{1}{N}$ .

Let us now prove (b). First note that  $0 \leq F_{p(N)} \leq E_N$  implies  $\|F_{p(N)}\| \leq \|E_N\| \leq 1$  for all  $N \in \mathbb{N}$ . By a routine approximation argument one shows that  $\|F_{p(N)} T - T\| \rightarrow 0$  for each  $T \in \mathbb{K}(X)$ , that is,  $(F_{p(N)})_N$  is an approximate unit for  $\mathbb{K}(X)$ .

Put  $F_N = \sum_{n=1}^N \theta_{x_n, x_n}$  for each  $N \in \mathbb{N}$ . Clearly,  $0 \leq F_N \leq F_{N+1}$  and, because of  $F_N \leq F_{p(N)} \leq E_N$ , we also have  $\|F_N\| \leq 1$ . By Lemma 2.7,  $(F_N)_N$  is an approximate unit for  $\mathbb{K}(X)$ . Proposition 2.3 now implies that  $(x_n)_n$  is a Parseval frame for  $X$ .  $\square$

We end this section by an example of a Parseval frame for  $\ell^2(A)$  (where  $A$  is an arbitrary  $\sigma$ -unital  $C^*$ -algebra) and the corresponding approximate

unit for  $\mathbb{K}(\ell^2(\mathbf{A}))$  that arises from Proposition 2.3. First we need a useful auxiliary result.

**Lemma 2.9.** *Let  $Y$  be a dense submodule of a Hilbert  $\mathbf{A}$ -module  $X$ . Suppose that a sequence  $(x_n)_n$  in  $X$  has the property*

$$A\langle y, y \rangle \leq \sum_{n=1}^{\infty} \langle y, x_n \rangle \langle x_n, y \rangle \leq B\langle y, y \rangle, \quad \forall y \in Y,$$

for some positive constants  $A$  and  $B$ . Then  $(x_n)_n$  is a frame for  $X$  with frame bounds  $A$  and  $B$ .

*Proof.* Let us define  $U_0 : Y \rightarrow \ell^2(\mathbf{A})$  by  $U_0 y = (\langle x_n, y \rangle)_n$ . Clearly,  $U_0$  is well defined,  $\mathbf{A}$ -linear, bounded and bounded from below. Let  $U : X \rightarrow \ell^2(\mathbf{A})$  be the continuation of  $U_0$ . Note that  $\|U\| = \|U_0\| \leq \sqrt{B}$ . Similarly, for  $x \in X$ , if  $(y_n)_n$  is a sequence in  $Y$  such that  $x = \lim_{n \rightarrow \infty} y_n$ , we have

$$\|Ux\| = \lim_{n \rightarrow \infty} \|U_0 y_n\| \geq \lim_{n \rightarrow \infty} \sqrt{A} \|y_n\| = \sqrt{A} \|x\|.$$

We now prove that  $U$  is an adjointable operator. First, put

$$U^*(a_1, \dots, a_N, 0, 0, \dots) = \sum_{n=1}^N x_n a_n, \quad \forall (a_1, \dots, a_N, 0, 0, \dots) \in c_{00}(\mathbf{A}).$$

By a routine verification one shows that

$$(8) \quad \langle Uy, z \rangle = \langle y, U^*z \rangle, \quad \forall y \in Y, \quad \forall z \in c_{00}(\mathbf{A}).$$

Suppose now that  $x = \lim_{n \rightarrow \infty} y_n$  with  $y_n \in Y$ . Then, by (8), we have  $\langle Uy_n, z \rangle = \langle y_n, U^*z \rangle$  for all  $n \in \mathbb{N}$  and  $z \in c_{00}(\mathbf{A})$ . By letting  $n \rightarrow \infty$  we obtain

$$(9) \quad \langle Ux, z \rangle = \langle x, U^*z \rangle, \quad \forall x \in X, \quad \forall z \in c_{00}(\mathbf{A}).$$

We now show that  $U^*$  is bounded on  $c_{00}(\mathbf{A})$ . Let  $z \in c_{00}(\mathbf{A})$ . Then

$$\begin{aligned} \|U^*z\| &= \sup\{\|\langle x, U^*z \rangle\| : x \in X, \|x\| \leq 1\} \\ &\stackrel{(9)}{=} \sup\{\|\langle Ux, z \rangle\| : x \in X, \|x\| \leq 1\} \\ &\leq \sqrt{B} \|z\|. \end{aligned}$$

This enables us to extend  $U^*$  by continuity to  $\ell^2(\mathbf{A})$ . It is now evident that (9) extends to the same equality that holds true for all  $x \in X$  and  $z \in \ell^2(\mathbf{A})$ . Thus,  $U$  is an adjointable operator. Since  $Ux = (\langle x_n, x \rangle)_n$  for all  $x \in X$  and  $A\|x\|^2 \leq \|Ux\|^2 \leq B\|x\|^2$  for all  $x \in X$ , it only remains to apply Theorem 2.6 from [2].  $\square$

*Example 2.10.* Let  $\mathbf{A}$  be a  $\sigma$ -unital  $C^*$ -algebra and let  $(e_n)_n$  be an increasing approximate unit for  $\mathbf{A}$ ; put additionally  $e_0 = 0$ . Let

$$f_n = (e_n - e_{n-1})^{\frac{1}{2}}, \quad n \in \mathbb{N}.$$

Consider  $\mathbf{A}$  as a Hilbert  $C^*$ -module over itself. Since

$$\sum_{n=1}^N \theta_{f_n, f_n} = \sum_{n=1}^N f_n f_n^* = \sum_{n=1}^N (e_n - e_{n-1}) = e_N, \quad \forall N \in \mathbb{N},$$

we conclude from Proposition 2.3 that  $(f_n)_n$  is a Parseval frame for  $\mathbf{A}$ .

For  $n, j \in \mathbb{N}$  consider the system

$$f_n^{(j)} = (0, \dots, 0, f_n, 0, \dots) \in \ell^2(\mathbf{A}) \quad (f_n \text{ on } j\text{-th position, } 0\text{'s elsewhere}).$$

We will show that the system  $(f_n^{(j)})_{n,j=1}^\infty$  is a Parseval frame for  $\ell^2(\mathbf{A})$ .

Let us first organize our system  $(f_n^{(j)})_{n,j=1}^\infty$  into a sequence. This can be done in a standard way by enumerating elements along finite diagonals of an infinite matrix starting from the upper left corner. Put

$$p(m) = \frac{1}{2}m(m+1), \quad m = 0, 1, 2, \dots$$

and observe that each natural number  $n$  can be written in a unique way as

$$n = p(m-1) + k_m, \quad m \in \mathbb{N}, \quad k_m \in \{1, 2, \dots, m\}.$$

We now put

$$x_n = x_{p(m-1)+k_m} = f_{m+1-k_m}^{(k_m)}, \quad n \in \mathbb{N}.$$

This gives us a sequence

$$f_1^{(1)}, f_2^{(1)}, f_1^{(2)}, f_3^{(1)}, f_2^{(2)}, f_1^{(3)}, f_4^{(1)}, f_3^{(2)}, f_2^{(3)}, f_1^{(4)}, \dots$$

Let us now show that

$$(10) \quad y = \sum_{n=1}^\infty x_n \langle x_n, y \rangle, \quad \forall y \in c_{00}(\mathbf{A}).$$

Fix an arbitrary  $y = (a_1, \dots, a_m, 0, 0, \dots) \in c_{00}(\mathbf{A})$ . Let  $\varepsilon > 0$ . Since  $(f_n)_n$  is a Parseval frame for  $\mathbf{A}$ , we can find  $N_0 \in \mathbb{N}$  such that

$$(11) \quad N \geq N_0 \Rightarrow \left\| a_i - \sum_{n=1}^N f_n \langle f_n, a_i \rangle \right\| < \frac{\varepsilon}{m}, \quad \forall i = 1, 2, \dots, m.$$

For such  $N_0$  consider  $p(N_0 + m)$ . We now claim that

$$(12) \quad N \geq p(N_0 + m) \Rightarrow \left\| y - \sum_{n=1}^N x_n \langle x_n, y \rangle \right\| < \varepsilon.$$

To show this, first observe that each term in the sum  $\sum_{n=1}^N x_n \langle x_n, y \rangle$  is of the form  $f_j^{(k)} \langle f_j^{(k)}, y \rangle$  which is in fact  $(f_j \langle f_j, a_k \rangle)^{(k)}$  - an element of  $\ell^2(\mathbf{A})$  with  $f_j \langle f_j, a_k \rangle$  on  $k$ -th position and 0's elsewhere.

Let us first prove (12) for  $N = p(N_0 + m)$ . Since in this case we have  $N = p(N_0 + m - 1) + (N_0 + m)$ , the last  $N_0 + m$  members among  $x_1, x_2, \dots, x_N$  are

$$f_{N_0+m}^{(1)}, \dots, f_{N_0+1}^{(m)}, \dots, f_1^{(N_0+m)}.$$



Thus

$$\begin{aligned}
\left\| y - \sum_{n=1}^N x_n \langle x_n, y \rangle \right\| &\leq \left\| a_1 - \sum_{n=1}^{N_0+m} f_n \langle f_n, a_1 \rangle \right\| + \left\| a_2 - \sum_{n=1}^{N_0+m-1} f_n \langle f_n, a_2 \rangle \right\| \\
&\quad + \dots + \left\| a_m - \sum_{n=1}^{N_0+1} f_n \langle f_n, a_m \rangle \right\| \\
&\stackrel{(11)}{<} \frac{\varepsilon}{m} + \frac{\varepsilon}{m} + \dots + \frac{\varepsilon}{m} = \varepsilon.
\end{aligned}$$

Next we prove (12) for  $N > p(N_0 + m)$ . By the same reasoning as above we get natural numbers  $N_1, N_2, \dots, N_m > N_0$  such that

$$\begin{aligned}
\left\| y - \sum_{n=1}^N x_n \langle x_n, y \rangle \right\| &\leq \left\| a_1 - \sum_{n=1}^{N_1} f_n \langle f_n, a_1 \rangle \right\| + \left\| a_2 - \sum_{n=1}^{N_2} f_n \langle f_n, a_2 \rangle \right\| \\
&\quad + \dots + \left\| a_m - \sum_{n=1}^{N_m} f_n \langle f_n, a_m \rangle \right\| \\
&\stackrel{(11)}{<} \frac{\varepsilon}{m} + \frac{\varepsilon}{m} + \dots + \frac{\varepsilon}{m} = \varepsilon.
\end{aligned}$$

This proves (10). In particular, by taking inner products by  $y$  in (10) we obtain

$$(13) \quad \langle y, y \rangle = \sum_{n=1}^{\infty} \langle y, x_n \rangle \langle x_n, y \rangle, \quad \forall y \in c_{00}(\mathbf{A}).$$

The desired conclusion, namely that  $(x_n)_n$  is a Parseval frame for  $\ell^2(\mathbf{A})$ , now follows directly from the preceding lemma.

Observe that the same construction can be done starting from an arbitrary Parseval frame  $(f_n)_n$  for  $\mathbf{A}$  - one can easily check that all the above arguments apply without changes.

By Proposition 2.3 we now know that the sequence  $(\sum_{n=1}^N \theta_{x_n, x_n})_N$  is an approximate unit for  $\mathbb{K}(\ell^2(\mathbf{A}))$ . Since each subsequence of an approximate unit is an approximate unit itself, we conclude that the sequence  $(\sum_{n=1}^{p(N)} \theta_{x_n, x_n})_N$  is also an approximate unit for  $\mathbb{K}(\ell^2(\mathbf{A}))$ .

Finally, note that the operators  $T_N = \sum_{n=1}^{p(N)} \theta_{x_n, x_n}$ ,  $N \in \mathbb{N}$ , are in fact of a very simple form. Indeed, by an easy computation one gets

$$T_N((a_n)_n) = \left( \sum_{n=1}^N f_n^2 a_1, \sum_{n=1}^{N-1} f_n^2 a_2, \dots, f_1^2 a_N, 0, 0, \dots \right), \quad \forall (a_n)_n \in \ell^2(\mathbf{A});$$

in other words,

$$T_N((a_n)_n) = (e_N a_1, e_{N-1} a_2, \dots, e_1 a_N, 0, 0, \dots), \quad \forall (a_n)_n \in \ell^2(\mathbf{A}).$$

## 3. OUTER FRAMES

Recall from the introduction that each frame  $(x_n)_n$  for a Hilbert  $A$ -module  $X$  gives rise to an adjointable surjection (namely, the corresponding synthesis operator) from  $\ell^2(A)$  to  $X$ . We open this section with the converse statement - a fact that is, although simple, of great importance in frame theory. We point out that here the underlying  $C^*$ -algebra  $A$  must be unital (cf. Example 1.7).

**Proposition 3.1.** *Let  $X$  be a Hilbert  $C^*$ -module over a unital  $C^*$ -algebra  $A$  and let  $T \in \mathbb{B}(\ell^2(A), X)$  be a surjection. Then there is a frame  $(x_n)_n$  for  $X$  whose synthesis operator is equal to  $T$ .*

*Proof.* Put  $x_n = Te^{(n)}$ ,  $n \in \mathbb{N}$ . By [2, Theorem 2.5], the sequence  $(x_n)_n$  is a frame for  $X$ . Denote the corresponding analysis operator by  $U$ . Then we have, for all  $x \in X$  and  $n \in \mathbb{N}$ ,

$$\langle U^*e^{(n)}, x \rangle = \langle e^{(n)}, Ux \rangle = \langle x_n, x \rangle = \langle Te^{(n)}, x \rangle,$$

which implies  $U^* = T$ .  $\square$

In order to obtain the non-unital version of Proposition 3.1, recall that each Hilbert  $C^*$ -module  $X$  over a non-unital  $C^*$ -algebra  $A$  can be regarded as a Hilbert  $C^*$ -module over a unital  $C^*$ -algebra  $\tilde{A}$ . Since frames for a Hilbert  $A$ -module  $X$  and frames for a Hilbert  $\tilde{A}$ -module  $X$  coincide, we conclude from Proposition 3.1 that each surjection in  $\mathbb{B}(\ell^2(\tilde{A}), X)$  serves as the synthesis operator of some frame for  $X$ .

In some situations this conclusion enables us to reduce the non-unital case to the unital one. However, as we shall see in the subsequent sections, this still does not resolve the difficulty observed in Example 1.7. Namely, there are surjections from  $\mathbb{B}(\ell^2(A), X)$  which cannot be extended to adjointable operators from  $\ell^2(\tilde{A})$  to  $X$  (which is precisely the case with the surjection  $T$  from Example 1.7). On the other hand, such surjections, as the same example indicates, might be associated with sequences that behave as frames and the only difference is that the members of such frame-like sequences need not belong to the original module  $X$ .

The preceding discussion suggests that our study of frames for Hilbert  $C^*$ -modules over non-unital  $C^*$ -algebras requires a more general setting. Thus, we shall extend our considerations to multiplier Hilbert  $C^*$ -modules.

To avoid unnecessary complications, we shall restrict ourselves in the analysis that follows to *infinite sequences*. At the end of this section we shall make appropriate comments on the corresponding results concerning finite frames.

First, in the remark that follows, we include for reader's convenience the most important facts concerning multiplier Hilbert  $C^*$ -modules (see [5] and [6]).

*Remark 3.2.* Let  $X$  be a Hilbert  $\mathbf{A}$ -module.

(a) There exists a Hilbert  $M(\mathbf{A})$ -module  $M(X)$  containing  $X$  as the ideal submodule associated with the ideal  $\mathbf{A}$  in  $M(\mathbf{A})$ ; i.e.,  $X = M(X)\mathbf{A}$ . It turns out that

$$X = \{x \in M(X) : \langle x, v \rangle \in \mathbf{A}, \forall v \in M(X)\}.$$

The extended module  $M(X)$  is called the multiplier module of  $X$ . It is known that  $M(X)$  can be naturally identified with  $\mathbb{B}(\mathbf{A}, X)$ . If  $\mathbf{A}$  is unital, or if  $X$  is AFG,  $M(X)$  coincides with  $X$ . For each  $v \in M(X)$  we have

$$\|v\| = \sup\{\|va\| : a \in \mathbf{A}, \|a\| \leq 1\} = \sup\{\|\langle v, x \rangle\| : x \in X, \|x\| \leq 1\}.$$

In particular, if  $\mathbf{A}$  is a  $C^*$ -algebra and if one takes  $X = A$ , then it turns out that the multiplier module  $M(X)$  coincides with  $M(\mathbf{A})$ .

(b) The strict topology on  $M(X)$  is locally convex topology generated by the family of seminorms  $v \mapsto \|va\|$ ,  $a \in \mathbf{A}$ , and  $v \mapsto \|\langle v, x \rangle\|$ ,  $x \in X$ . The multiplier module  $M(X)$  is complete with respect to the strict topology. If  $(e_\lambda)_\lambda$  is an approximate unit for  $\mathbf{A}$ , then each  $v \in M(X)$  satisfies  $v = (\text{strict}) \lim_\lambda ve_\lambda$ . Hence,  $X$  is strictly dense in  $M(X)$ . In fact,  $M(X)$  is the strict completion of  $X$ .

(c) For the generalized Hilbert space  $\ell^2(\mathbf{A})$  over  $\mathbf{A}$  we get

$$M(\ell^2(\mathbf{A})) = \left\{ (c_n)_n \in M(\mathbf{A})^\mathbb{N} : \sum_{n=1}^{\infty} c_n^* c_n \text{ converges strictly} \right\}$$

and the  $M(\mathbf{A})$ -valued inner product on  $M(\ell^2(\mathbf{A}))$  is given by

$$\langle (c_n)_n, (d_n)_n \rangle = (\text{strict}) \sum_{n=1}^{\infty} c_n^* d_n.$$

The set  $c_{00}(M(\mathbf{A}))$  of all finite sequences of elements of  $M(\mathbf{A})$  is strictly dense in  $M(\ell^2(\mathbf{A}))$ .

(d) If  $Y$  is a Hilbert  $\mathbf{A}$ -module, each operator  $T \in \mathbb{B}(X, Y)$  has an extension  $T_M \in \mathbb{B}(M(X), M(Y))$ . The extended operator  $T_M$  is obtained as the strict continuation of  $T$ ; hence, it is uniquely determined. The map  $T \mapsto T_M$  is a bijection of  $\mathbb{B}(X, Y)$  and  $\mathbb{B}(M(X), M(Y))$  such that  $\|T_M\| = \|T\|$  and  $(T_M)^* = (T^*)_M$  for all  $T$  in  $\mathbb{B}(X, Y)$ .

We now introduce the concept of an outer frame for Hilbert  $C^*$ -modules. In comparison with frames for  $X$  the difference is that the elements of an outer frame for  $X$  are merely members of a larger module  $M(X)$  and need not belong to  $X$ .

*Definition 3.3.* Let  $X$  be a Hilbert  $C^*$ -module. A sequence  $(v_n)_n$  in  $M(X)$  is called an *outer frame* for  $X$  if  $v_n \in M(X) \setminus X$  for at least one  $n \in \mathbb{N}$ , and

if there exist positive constants  $A$  and  $B$  such that

$$(14) \quad A\langle x, x \rangle \leq \sum_{n=1}^{\infty} \langle x, v_n \rangle \langle v_n, x \rangle \leq B\langle x, x \rangle, \quad \forall x \in X,$$

where the series  $\sum_{n=1}^{\infty} \langle x, v_n \rangle \langle v_n, x \rangle$  converges in norm of  $\mathbf{A}$ .

If  $A = B = 1$ , the sequence  $(v_n)_n$  is called an *outer Parseval frame* for  $X$ .

A sequence  $(v_n)_n$  is said to be an *outer Bessel sequence* if only the second inequality in (14) is satisfied.

Notice that each  $\langle v_n, x \rangle$  belongs to  $\mathbf{A}$  for every  $x \in X$ , even for those  $n$  for which  $v_n \in M(X) \setminus X$ ; this is a consequence of Remark 3.2(a).

We also note that outer Parseval frames (though, not under that name) appeared already in [20] in the context of a generalized version of Kasparov's stabilization theorem.

*Remark 3.4.* By definition, outer frames do not exist if  $X$  is strictly complete, i.e., if  $M(X) = X$  (by Remark 3.2(a), this is the case when  $A$  is unital, or when  $X$  is AFG).

If  $X$  is a countably generated Hilbert  $\mathbf{A}$ -module such that  $M(X) \neq X$  then outer frames exist in abundance. To obtain an outer frame for  $X$  we can simply add any vector from  $M(X) \setminus X$  to an arbitrary frame for  $X$ .

Let us now show that the sequence from Example 1.7 is an outer frame.

*Example 3.5.* Let us keep the notations from Example 1.7. We have seen that  $\lim_{N \rightarrow \infty} \|a - \sum_{n=1}^N p_n a\| = 0$  for each  $a \in \mathbb{K}(H)$ . This conclusion can be rewritten in the modular context in the form

$$a = \lim_{N \rightarrow \infty} \sum_{n=1}^N p_n a = \lim_{N \rightarrow \infty} \sum_{n=1}^N s_n \langle s_n, a \rangle = \sum_{n=1}^{\infty} s_n \langle s_n, a \rangle,$$

with the norm convergence of the series at the end (recall that the norm on the Hilbert  $\mathbb{K}(H)$ -module  $\mathbb{K}(H)$  coincides with the original, i.e., operator norm on  $\mathbb{K}(H)$ ). By taking the inner product of both sides by  $a$  we get

$$\langle a, a \rangle = \sum_{n=1}^{\infty} \langle a, s_n \rangle \langle s_n, a \rangle, \quad \forall a \in \mathbb{K}(H).$$

Thus,  $(s_n)_n$  is, being a sequence in  $\mathbb{B}(H) \setminus \mathbb{K}(H)$ , an outer Parseval frame for  $\mathbb{K}(H)$ .

We begin our study of outer frames by introducing their analysis and synthesis operators. It turns out that these operators have the same properties as the corresponding operators for frames.

**Proposition 3.6.** *Let  $(v_n)_n$  be an outer frame for a Hilbert  $\mathbf{A}$ -module  $X$ . Then its analysis operator*

$$U : X \rightarrow \ell^2(\mathbf{A}), \quad U(x) = (\langle v_n, x \rangle)_n,$$

is well defined, adjointable and bounded from below. The synthesis operator  $U^*$  is surjective and satisfies

$$U^*((a_n)_n) = \sum_{n=1}^{\infty} v_n a_n, \quad \forall (a_n)_n \in \ell^2(A),$$

where this series converges in norm.

*Proof.* By defining inequalities (14), the operator  $U$  is well defined,  $A$ -linear, bounded by  $\sqrt{B}$ , and bounded from below by  $\sqrt{A}$ . Let us show that  $U$  is an adjointable operator. For  $N \in \mathbb{N}$  and any  $y = (a_1, \dots, a_N, 0, \dots) \in c_{00}(A)$ , we put

$$U^*((a_1, \dots, a_N, 0, \dots)) = \sum_{n=1}^N v_n a_n.$$

Observe that all  $v_n a_n$  belong to  $X$  since  $M(X)A = X$  (see Remark 3.2(a)). By a routine verification one concludes that

$$(15) \quad \langle U^* y, x \rangle = \langle y, Ux \rangle, \quad \forall x \in X, \quad \forall y \in c_{00}(A).$$

We now claim that  $U^*$  is bounded on  $c_{00}(A)$ . Indeed, we have for each  $y \in c_{00}(A)$

$$\begin{aligned} \|U^* y\| &= \sup\{\|\langle U^* y, x \rangle\| : x \in X, \|x\| \leq 1\} \\ &\stackrel{(15)}{=} \sup\{\|\langle y, Ux \rangle\| : x \in X, \|x\| \leq 1\} \\ &\leq \sqrt{B} \|y\|. \end{aligned}$$

This enables us to extend  $U^*$  to all of  $\ell^2(A)$  by continuity. Moreover, one easily concludes that equality (15) extends then to

$$\langle U^* y, x \rangle = \langle y, Ux \rangle, \quad \forall x \in X, \quad \forall y \in \ell^2(A).$$

This proves that  $U$  is an adjointable operator. The preceding discussion also shows that  $U^*$  is given by  $U^*((a_n)_n) = \sum_{n=1}^{\infty} v_n a_n$  for all  $(a_n)_n \in \ell^2(A)$ . Since  $U$  is bounded from below,  $U^*$  is surjective.  $\square$

An immediate consequence of (the proof of) the preceding proposition is the corresponding statement concerning outer Bessel sequences.

**Corollary 3.7.** *Let  $(v_n)_n$  be an outer Bessel sequence for a Hilbert  $A$ -module  $X$ . Then its analysis operator*

$$U : X \rightarrow \ell^2(A), \quad U(x) = (\langle v_n, x \rangle)_n,$$

*is a well defined adjointable operator. The synthesis operator  $U^*$  satisfies*

$$U^*((a_n)_n) = \sum_{n=1}^{\infty} v_n a_n, \quad \forall (a_n)_n \in \ell^2(A),$$

*where this series converges in norm.*

As we shall see, outer frames for a countably generated Hilbert  $\mathbf{A}$ -module  $X$  are exactly what one should add to the set of all frames for  $X$  in order to establish a bijective correspondence with surjections from  $\mathbb{B}(\ell^2(\mathbf{A}), X)$ . To do that, we need a unified approach to frames and outer frames, and it turns out that this can be done by using another new concept: strict frames for multiplier Hilbert  $C^*$ -modules.

*Definition 3.8.* Let  $X$  be a Hilbert  $\mathbf{A}$ -module. A sequence  $(v_n)_n$  in the multiplier module  $M(X)$  is called a *strict frame* for  $M(X)$  if there exist positive constants  $A$  and  $B$  such that

$$(16) \quad A\langle v, v \rangle \leq (\text{strict}) \sum_{n=1}^{\infty} \langle v, v_n \rangle \langle v_n, v \rangle \leq B\langle v, v \rangle, \quad \forall v \in M(X).$$

If  $A = B = 1$ , i.e., if

$$(17) \quad (\text{strict}) \sum_{n=1}^{\infty} \langle v, v_n \rangle \langle v_n, v \rangle = \langle v, v \rangle, \quad \forall v \in M(X),$$

the sequence  $(v_n)_n$  is called a *strict Parseval frame* for  $M(X)$ .

*Example 3.9.* Let  $\mathbf{A}$  be a non-unital  $C^*$ -algebra. Then the sequence  $(e^{(n)})_n$  is a strict Parseval frame for the multiplier module  $M(\ell^2(\mathbf{A}))$ . This follows immediately from Remark 3.2(c).

**Proposition 3.10.** *Let  $X$  be a Hilbert  $\mathbf{A}$ -module. Every strict frame for  $M(X)$  is a frame or an outer frame for  $X$ .*

*Proof.* Let  $(v_n)_n$  be a strict frame for  $M(X)$ . By definition of the strict convergence in  $M(\mathbf{A})$  this implies that the series  $\sum_{n=1}^{\infty} \langle v, v_n \rangle \langle v_n, v \rangle a$  is norm convergent in  $\mathbf{A}$  for all  $a \in \mathbf{A}$ . Then the series  $\sum_{n=1}^{\infty} \langle va, v_n \rangle \langle v_n, va \rangle$  is norm convergent for all  $v \in M(X)$  and  $a \in \mathbf{A}$ . Since, by Proposition 2.31 from [21], each  $x \in X$  can be written in the form  $x = va$  for some  $v \in X$  and  $a \in \mathbf{A}$ , the preceding discussion shows that the series  $\sum_{n=1}^{\infty} \langle x, v_n \rangle \langle v_n, x \rangle$  converges in norm for every  $x \in X$ . Now, if each  $v_n$  belongs to  $X$  then  $(v_n)_n$  is a frame for  $X$ , and if some  $v_n$  is in  $M(X) \setminus X$  then  $(v_n)_n$  is an outer frame for  $X$ .  $\square$

*Remark 3.11.* If  $X$  is a strictly complete Hilbert  $C^*$ -module, i.e., if  $M(X) = X$  (for example, when  $\mathbf{A}$  is unital or  $X$  is AFG), the preceding proposition implies that strict frames are simply frames for  $X$ .

We begin our study of strict frames by showing that the conditions in the definition of a strict frame can be relaxed in a manner similar to that in Theorem 2.2.

**Theorem 3.12.** *Let  $X$  be a Hilbert  $\mathbf{A}$ -module and let  $(v_n)_n$  be a sequence in  $M(X)$ . Then the following two conditions are equivalent:*

- (a)  $(v_n)_n$  is a strict frame for  $M(X)$ .
- (b) The series  $\sum_{n=1}^{\infty} \langle v, v_n \rangle \langle v_n, v \rangle$  converges strictly for all  $v$  in  $M(X)$  and there is  $A > 0$  such that  $A\|v\|^2 \leq \|(\text{strict}) \sum_{n=1}^{\infty} \langle v, v_n \rangle \langle v_n, v \rangle\|$  for all  $v$  in  $M(X)$ .

If  $(v_n)_n$  is a strict frame for  $M(X)$ , its analysis operator

$$(18) \quad U : M(X) \rightarrow M(\ell^2(A)), \quad U(v) = (\langle v_n, v \rangle)_n,$$

is well defined, adjointable and bounded from below. The synthesis operator  $U^*$  is surjective and satisfies

$$(19) \quad U^*((b_n)_n) = (\text{strict}) \sum_{n=1}^{\infty} v_n b_n, \quad \forall (b_n)_n \in M(\ell^2(A)).$$

In particular,  $v_n = U^*e^{(n)}$  for all  $n \in \mathbb{N}$ .

*Proof.* Let us first make an observation concerning elements of  $M(\ell^2(A))$ . For each  $(b_n)_n \in M(\ell^2(A))$  we know that  $b := (\text{strict}) \sum_{n=1}^{\infty} b_n^* b_n$  exists, which means that for all  $a \in A$  the series  $\sum_{n=1}^{\infty} a b_n^* b_n$  and  $\sum_{n=1}^{\infty} b_n^* b_n a$  converge in norm to  $ab$  and  $ba$ , respectively. In particular, if we assume that  $A$  is faithfully and non-degenerately represented on some Hilbert space  $H$ , then the series  $\sum_{n=1}^{\infty} b_n^* b_n$  also converges to  $b$  in the strong operator topology. This, in particular, implies that  $\sum_{n=1}^N b_n^* b_n \leq b$  and, consequently,  $\|\sum_{n=1}^N b_n^* b_n\| \leq \|b\|$  for all  $N \in \mathbb{N}$ .

Let us now assume (b).

By the first assumption in (b), the operator  $U : M(X) \rightarrow M(\ell^2(A))$ ,  $U(v) = (\langle v_n, v \rangle)_n$ , is well defined and  $M(A)$ -linear. By applying the closed graph theorem, precisely as in the first part of the proof of Theorem 2.1, one shows that  $U$  is bounded. Put  $\|U\|^2 = B$ .

As in the proof of Proposition 3.10 we observe that  $Ux \in \ell^2(A)$  for each  $x \in X$ . Thus, the restriction  $U_X$  of  $U$  to  $X$  takes values in  $\ell^2(A)$ . Since norms on  $M(X)$  and  $M(\ell^2(A))$  extend the original norms on  $X$  and  $\ell^2(A)$ , respectively, we also have  $\|U_X x\| \leq \sqrt{B}\|x\|$  for all  $x \in X$ .

We now prove that  $U$  is an adjointable operator. Let us first define  $U^*$  on finite sequences by putting  $U^*(b_1, \dots, b_N, 0, \dots) = \sum_{n=1}^N v_n b_n$  for each  $(b_1, \dots, b_N, 0, \dots) \in c_{00}(M(A))$ . In particular, we have  $U^*e^{(n)} = v_n$ ,  $n \in \mathbb{N}$ . By a routine computation one finds

$$(20) \quad \langle z, Uv \rangle = \langle U^*z, v \rangle, \quad \forall z \in c_{00}(M(A)), \quad \forall v \in M(X).$$

We now claim that  $U^*$  is bounded on  $c_{00}(M(A))$ . Indeed, we have for each  $z \in c_{00}(M(A))$

$$\begin{aligned} \|U^*z\| &= \sup\{\|\langle U^*z, v \rangle\| : v \in M(X), \|v\| \leq 1\} \\ &\stackrel{(20)}{=} \sup\{\|\langle z, Uv \rangle\| : v \in M(X), \|v\| \leq 1\} \\ &\leq \sqrt{B}\|z\|. \end{aligned}$$

Next we claim: if we have  $z \in M(\ell^2(\mathbf{A}))$  and a net  $(z_\lambda)_\lambda$  in  $c_{00}(M(\mathbf{A}))$  such that  $z = (\text{strict}) \lim_\lambda z_\lambda$ , then there exists  $(\text{strict}) \lim_\lambda U^* z_\lambda$  in  $M(X)$ . By Remark 3.2(b) it is enough to prove that  $(U^* z_\lambda)_\lambda$  is a strictly Cauchy net. This means that  $((U^* z_\lambda)a)_\lambda$  and  $(\langle U^* z_\lambda, x \rangle)_\lambda$  should be Cauchy nets, for all  $a \in \mathbf{A}$  and  $x \in X$ .

First, for each  $a \in \mathbf{A}$  and  $\lambda, \mu$ , we have

$$\|(U^* z_\mu)a - (U^* z_\lambda)a\| = \|U^*(z_\mu a - z_\lambda a)\| \leq \sqrt{B} \|z_\mu a - z_\lambda a\|.$$

This is enough, since  $(z_\lambda a)_\lambda$  is a norm convergent net.

Secondly, for each  $x \in X$ , we have  $\langle U^* z_\lambda, x \rangle = \langle z_\lambda, Ux \rangle$ ; but  $(\langle z_\lambda, Ux \rangle)_\lambda$  is a convergent net since  $Ux \in \ell^2(\mathbf{A})$ .

Let us now fix an arbitrary  $z = (b_1, b_2, b_3, \dots) \in M(\ell^2(\mathbf{A}))$ . By Remark 3.2(c) we have  $z = (\text{strict}) \lim_{N \rightarrow \infty} z_N$  with  $z_N = \sum_{n=1}^N e^{(n)} b_n$ . By the preceding paragraph there exists  $(\text{strict}) \lim_{N \rightarrow \infty} U^* z_N$  in  $M(X)$  and we denote this limit by  $U^* z$ .

Let us now prove that  $\langle z, Uv \rangle = \langle U^* z, v \rangle$  for all  $v \in M(X)$ .

First, for each  $a \in \mathbf{A}$  we know by (20) that  $a \langle z_N, Uv \rangle = a \langle U^* z_N, v \rangle$  for all  $v \in M(X)$ . This implies

$$(21) \quad \langle z_N a^*, Uv \rangle = \langle (U^* z_N) a^*, v \rangle, \quad \forall a \in \mathbf{A}, \quad \forall v \in M(X).$$

Since  $\|z_N a^* - z a^*\| \rightarrow 0$  as  $N$  tends to infinity, the left hand side in (21) converges to  $\langle z a^*, Uv \rangle$ . On the other hand,  $\|(U^* z_N) a^* - (U^* z) a^*\| \rightarrow 0$  as  $N$  tends to infinity; hence the right hand side in (21) converges to  $\langle (U^* z) a^*, v \rangle$ . Thus, by letting  $N \rightarrow \infty$  in (21), we obtain

$$a \langle z, Uv \rangle = a \langle U^* z, v \rangle, \quad \forall a \in \mathbf{A}, \quad \forall v \in M(X)$$

or, equivalently,

$$a(\langle z, Uv \rangle - \langle U^* z, v \rangle) = 0, \quad \forall a \in \mathbf{A}, \quad \forall v \in M(X).$$

This is enough to conclude  $\langle z, Uv \rangle = \langle U^* z, v \rangle$  for each  $v \in M(X)$ . As  $z$  was arbitrary element of  $M(\ell^2(\mathbf{A}))$ , we have finally proved that  $U$  is an adjointable operator. Recall that  $U^*$  is given by

$$U^*(b_1, b_2, b_3, \dots) = (\text{strict}) \lim_{N \rightarrow \infty} U^* \left( \sum_{n=1}^N e^{(n)} b_n \right) = (\text{strict}) \lim_{N \rightarrow \infty} \sum_{n=1}^N v_n b_n$$

for each  $(b_1, b_2, b_3, \dots) \in M(\ell^2(\mathbf{A}))$ . In other words,

$$U^*(b_1, b_2, b_3, \dots) = (\text{strict}) \sum_{n=1}^{\infty} v_n b_n, \quad \forall (b_1, b_2, b_3, \dots) \in M(\ell^2(\mathbf{A})).$$

Furthermore,  $U^*$  is, being adjointable, norm-continuous. Since for each  $(a_1, a_2, a_3, \dots) \in \ell^2(\mathbf{A})$  we have  $(a_1, a_2, a_3, \dots) = \lim_{N \rightarrow \infty} \sum_{n=1}^N e^{(n)} a_n$  with convergence in norm, this gives us  $U^*(a_1, a_2, a_3, \dots) = \sum_{n=1}^{\infty} v_n a_n$ , where this series converges with respect to the norm. Hence,  $U_X : X \rightarrow \ell^2(\mathbf{A})$  is also an adjointable operator and we have inequalities

$$A\|v\|^2 \leq \|Uv\|^2 \leq B\|v\|^2, \quad \forall v \in M(X),$$



$$A\|x\|^2 \leq \|U_X x\|^2 \leq B\|x\|^2, \quad \forall x \in X.$$

Proposition 2.1 from [2] now implies

$$A\langle v, v \rangle \leq (\text{strict}) \sum_{n=1}^{\infty} \langle v, v_n \rangle \langle v_n, v \rangle \leq B\langle v, v \rangle, \quad \forall v \in M(X),$$

$$A\langle x, x \rangle \leq \sum_{n=1}^{\infty} \langle x, v_n \rangle \langle v_n, x \rangle \leq B\langle x, x \rangle, \quad \forall x \in X.$$

In particular, since  $U$  and  $U_X$  are bounded from below,  $U^*$  and  $(U_X)^*$  are surjective.  $\square$

**Proposition 3.13.** *Let  $X$  and  $Y$  be Hilbert  $C^*$ -modules and  $T \in \mathbb{B}(M(X), M(Y))$ . Then  $T$  maps strict frames for  $M(X)$  to strict frames for  $M(Y)$  if and only if  $T$  is surjective.*

*Proof.* Let  $(v_n)_n$  be a strict frame for  $M(X)$  and  $T \in \mathbb{B}(M(X), M(Y))$  a surjective operator. Denote by  $A$  and  $B$  the frame bounds of  $(v_n)_n$ . Since  $T$  is a surjection,  $T^*$  is bounded from below, so there exists  $m > 0$  such that  $\|T^*w\| \geq m\|w\|$  for all  $w \in M(Y)$ . Let  $w_n = Tv_n$ ,  $n \in \mathbb{N}$ . Observe that, for each  $w \in M(Y)$ , we have  $\langle w, w_n \rangle \langle w_n, w \rangle = \langle T^*w, v_n \rangle \langle v_n, T^*w \rangle$ . Therefore, there exists (strict)  $\sum_{n=1}^{\infty} \langle w, w_n \rangle \langle w_n, w \rangle$ . Moreover, for each  $w \in M(Y)$ ,

$$\begin{aligned} \left\| (\text{strict}) \sum_{n=1}^{\infty} \langle w, w_n \rangle \langle w_n, w \rangle \right\| &= \left\| (\text{strict}) \sum_{n=1}^{\infty} \langle T^*w, v_n \rangle \langle v_n, T^*w \rangle \right\| \\ &\geq A \| \langle T^*w, T^*w \rangle \| \\ &= A \| T^*w \|^2 \\ &\geq Am^2 \| w \|^2. \end{aligned}$$

By Theorem 3.12,  $(w_n)_n$  is a strict frame for  $M(Y)$ .

Conversely, suppose that  $T \in \mathbb{B}(M(X), M(Y))$  preserves strict frames. So, if  $(v_n)_n$  is a strict frame for  $M(X)$ , then  $(Tv_n)_n$  is a strict frame for  $M(Y)$ . If  $U$  is the analysis operator for  $(v_n)_n$ , then  $UT^*$  is the analysis operator for  $(Tv_n)_n$ , so  $TU^*$  is the corresponding synthesis operator. By Theorem 3.12,  $TU^*$  is surjective; hence,  $T$  must be surjective.  $\square$

As a direct consequence we now get an analog of Proposition 3.1 for strict frames. Note that here we do not need any assumptions on the underlying  $C^*$ -algebra  $A$ . In fact, when  $A$  is unital the statement of the following corollary reduces, due to Remark 3.11 and Remark 3.2(a), to Proposition 3.1.

**Corollary 3.14.** *If  $T \in \mathbb{B}(M(\ell^2(A)), M(X))$  is a surjection, then there exists a unique strict frame for  $M(X)$  whose synthesis operator is equal to  $T$ .*

*Proof.* Since  $(e^{(n)})_n$  is a strict frame for  $M(\ell^2(\mathbf{A}))$  (see Example 3.9), the preceding proposition implies that  $(v_n)_n$  defined by  $v_n = Te^{(n)}$ ,  $n \in \mathbb{N}$ , is a strict frame for  $M(X)$ . Its synthesis operator  $U^*$  also satisfies, by the last assertion of Theorem 3.12,  $U^*e^{(n)} = v_n$  for all  $n \in \mathbb{N}$ . This implies that  $U^*$  and  $T$  coincide on  $c_{00}(\mathbf{A})$ . Since  $c_{00}(\mathbf{A})$  is strictly dense in  $M(\ell^2(\mathbf{A}))$  and both  $U^*$  and  $T$  are strictly continuous (see Remark 3.2(d)), this is enough to conclude that  $U^* = T$ . Uniqueness is evident.  $\square$

Next we show the reconstruction property of strict frames.

Let  $X$  be a Hilbert  $\mathbf{A}$ -module. Suppose that  $(v_n)_n$  is a strict frame for  $M(X)$  with the analysis operator  $U \in \mathbb{B}(M(X), M(\ell^2(\mathbf{A})))$ . Then  $U^*U$  is an invertible operator for which we have

$$U^*Uy = (\text{strict}) \sum_{n=1}^{\infty} v_n \langle v_n, y \rangle, \quad \forall y \in M(X).$$

If we put  $U^*Uy = v$ , this can be rewritten as

$$v = (\text{strict}) \sum_{n=1}^{\infty} v_n \langle v_n, (U^*U)^{-1}v \rangle = (\text{strict}) \sum_{n=1}^{\infty} v_n \langle (U^*U)^{-1}v_n, v \rangle$$

for all  $v \in M(X)$ . Let  $w_n = (U^*U)^{-1}v_n$ ,  $n \in \mathbb{N}$ . By the preceding corollary  $(w_n)_n$  is also a strict frame for  $M(X)$  that satisfies

$$(22) \quad v = (\text{strict}) \sum_{n=1}^{\infty} v_n \langle w_n, v \rangle, \quad \forall v \in M(X).$$

If we denote by  $V$  the analysis operator of  $(w_n)_n$  then the above equality can be rewritten as  $U^*V = I$ . This obviously implies  $V^*U = I$ , so we also have

$$(23) \quad v = (\text{strict}) \sum_{n=1}^{\infty} w_n \langle v_n, v \rangle, \quad \forall v \in M(X).$$

By the last part of the proof of Theorem 3.12 the above two equalities give us, with the respect to the norm topology on  $X$ ,

$$(24) \quad x = \sum_{n=1}^{\infty} v_n \langle w_n, x \rangle = \sum_{n=1}^{\infty} w_n \langle v_n, x \rangle, \quad \forall x \in X.$$

In particular, if  $(v_n)_n$  is a strict Parseval frame for  $M(X)$ , the above equalities reduce to

$$(25) \quad v = (\text{strict}) \sum_{n=1}^{\infty} v_n \langle v_n, v \rangle, \quad \forall v \in M(X),$$

and

$$(26) \quad x = \sum_{n=1}^{\infty} v_n \langle v_n, x \rangle, \quad \forall x \in X.$$

*Remark 3.15.* If  $(x_n)_n$  is a frame for  $X$ , the reconstruction property from Theorem 1.2 shows that  $X$  is countably generated. Similarly, if  $(v_n)_n$  is a strict frame for  $M(X)$ , the reconstruction formula (24) shows that  $X$  is countably generated by  $v_n$ 's and the only difference is that here the generators (frame members) are elements of  $M(X)$  and need not belong to  $X$ . This property is introduced and discussed in [20]. By Definition 2.1 from [20], a Hilbert  $A$ -module  $X$  is countably generated in  $M(X)$  if there exists a sequence  $(v_n)_n$  in  $M(X)$  such that the set  $\text{span}\{v_n a : n \in \mathbb{N}, a \in A\}$  is norm-dense in  $X$ . It is proved in [20] that each Hilbert  $C^*$ -module  $X$  that is countably generated in  $M(X)$  possesses a Parseval frame or an outer Parseval frame. Finally, the reconstruction property (26) for such frames is derived in Theorem 3.4 in [20].

*Remark 3.16.* Having obtained the reconstruction formulae for strict frames (in particular, (26)), we can now generalize the statement of Proposition 2.3 in the following way: Let  $X$  be a Hilbert  $C^*$ -module. Then a sequence  $(v_n)_n$  of elements of  $M(X)$  is a strict Parseval frame for  $X$  if and only if the sequence  $(\sum_{n=1}^N \theta_{v_n, v_n})_N$  has the property  $T = \lim_{N \rightarrow \infty} T(\sum_{n=1}^N \theta_{v_n, v_n})$  for each  $T$  in  $\mathbb{K}(X)$ .

The proof is in fact the same as that of Proposition 2.3, so we omit the details.

We are now ready to extend Proposition 3.1 to the non-unital case. To do that, we first show that the class of strict frames for the multiplier module  $M(X)$  consists precisely of all frames and outer frames for  $X$ . Let us start with the following technical result.

**Lemma 3.17.** *Let  $X$  and  $Y$  be Hilbert  $A$ -modules, and  $T \in \mathbb{B}(X, Y)$ . Let  $T_M \in \mathbb{B}(M(X), M(Y))$  be the strict extension of  $T$ .*

(a) *If  $A$  and  $B$  are some positive constants, then*

$$(27) \quad A\|x\|^2 \leq \|Tx\|^2 \leq B\|x\|^2, \quad \forall x \in X,$$

*if and only if it*

$$(28) \quad A\|v\|^2 \leq \|T_M v\|^2 \leq B\|v\|^2, \quad \forall v \in M(X).$$

(b)  *$T$  is a surjection if and only if  $T_M$  is a surjection.*

*Proof.* Let us prove (a). If  $M(X) = X$ , there is nothing to prove. Hence, we assume that  $M(X) \neq X$ . Obviously, (28) implies (27), so we only need to prove the converse. First we recall a useful result from Remark 3.2(a), namely

$$(29) \quad \|v\| = \sup\{\|va\| : a \in A, \|a\| \leq 1\}, \quad \forall v \in M(X).$$

By applying this formula to  $U_M v \in M(Y)$  for  $v \in M(X)$ , we get

$$\begin{aligned} \|U_M v\| &= \sup \{ \|(U_M v)a\| : a \in \mathbf{A}, \|a\| \leq 1 \} \\ &= \sup \{ \|U_M(va)\| : a \in \mathbf{A}, \|a\| \leq 1 \} \\ &\quad (\text{since } va \in X) \\ &= \sup \{ \|U(va)\| : a \in \mathbf{A}, \|a\| \leq 1 \}. \end{aligned}$$

Using the first inequality from the hypothesis we get

$$\|U_M v\| \geq \sqrt{A} \sup \{ \|va\| : a \in \mathbf{A}, \|a\| \leq 1 \} \stackrel{(29)}{=} \sqrt{A} \|v\|,$$

while the second inequality from the hypothesis gives us

$$\|U_M v\| \leq \sqrt{B} \sup \{ \|va\| : a \in \mathbf{A}, \|a\| \leq 1 \} \stackrel{(29)}{=} \sqrt{B} \|v\|.$$

To prove (b) suppose first that  $T$  is a surjection. If  $M(Y) = Y$  then, trivially,  $T_M$  is a surjection. So, let us assume that  $M(Y) \neq Y$ , observe that  $T^*$  is bounded from below and hence, by (a), that  $(T^*)_M$  is also bounded from below. Recall from Remark 3.2(d) that  $(T^*)_M = (T_M)^*$ . By the preceding conclusion we know that  $(T_M)^*$  is bounded from below; thus,  $T_M$  is a surjection. Suppose now that  $T_M$  is a surjection. Then  $(T_M)^* = (T^*)_M$  is bounded from below. By (a),  $T^*$  is also bounded from below which implies that  $T$  is a surjection.  $\square$

**Theorem 3.18.** *Let  $X$  be a Hilbert  $\mathbf{A}$ -module and  $(x_n)_n$  a sequence in  $M(X)$ . Then  $(x_n)_n$  is a strict frame for  $M(X)$  if and only if  $(x_n)_n$  is a frame or an outer frame for  $X$ .*

*Proof.* If  $M(X) = X$  then, by Remarks 3.4 and 3.11, there is nothing to prove. So, let us assume that  $M(X) \neq X$ . One direction is proved in Proposition 3.10.

Suppose  $(x_n)_n$  is a frame or an outer frame for  $X$  with frame bounds  $A$  and  $B$ . Denote by  $U \in \mathbb{B}(X, \ell^2(\mathbf{A}))$  the corresponding analysis operator. Then  $U$  is bounded by  $\sqrt{B}$  and bounded from below by  $\sqrt{A}$  - if  $(x_n)_n$  is a frame this is already observed in the introduction, and if  $(x_n)_n$  is an outer frame Proposition 3.6 applies. So,

$$A\|x\|^2 \leq \|Ux\|^2 \leq B\|x\|^2, \quad \forall x \in X.$$

Let us now consider the extended operators  $U_M \in \mathbb{B}(M(X), M(\ell^2(\mathbf{A})))$  and  $(U^*)_M \in \mathbb{B}(M(\ell^2(\mathbf{A})), M(X))$ . By Lemma 3.17 we now know that

$$A\|v\|^2 \leq \|U_M v\|^2 \leq B\|v\|^2, \quad \forall v \in M(X),$$

and  $(U_M)^* = (U^*)_M$  is a surjection. Observe that  $(U^*)_M e^{(n)} = x_n$  for all  $n \in \mathbb{N}$ . By Proposition 3.13,  $(x_n)_n$  is a strict frame for  $M(X)$ .  $\square$

We are now in position to prove the key result of this section.

**Theorem 3.19.** *Let  $X$  be a Hilbert  $A$ -module and let  $T \in \mathbb{B}(\ell^2(A), X)$  be a surjection. Then there exists a unique frame or outer frame  $(x_n)_n$  for  $X$  whose synthesis operator coincides with  $T$ .*

*Proof.* If  $A$  is unital, Proposition 3.1 applies.

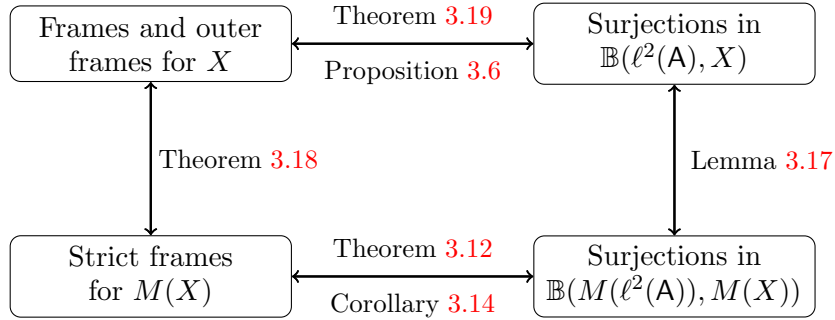
Assume that  $A$  is non-unital. By Lemma 3.17,  $T_M \in \mathbb{B}(M(\ell^2(A)), M(X))$  is also a surjection. By Corollary 3.14 the sequence  $(v_n)_n$  defined by  $v_n = T_M e^{(n)}$ ,  $n \in \mathbb{N}$ , is a strict frame for  $M(X)$  whose synthesis operator is equal to  $T_M$ . By Proposition 3.10,  $(v_n)_n$  is a frame or an outer frame for  $X$  depending on whether all  $v_n$ 's belong to  $X$  or not. Denote by  $U$  the corresponding analysis operator.

By definitions of a frame and an outer frame, we have that  $(\langle v_n, x \rangle)_n \in \ell^2(A)$  for all  $x \in X$ . By Proposition 3.6 (and the corresponding property of frames observed in the introduction) we know that  $U^*((a_n)_n) = \sum_{n=1}^{\infty} v_n a_n$ , where this series converges in norm of  $X$  for all  $(a_n)_n \in \ell^2(A)$ . It follows from (19) that

$$T((a_n)_n) = T_M((a_n)_n) = \sum_{n=1}^{\infty} v_n a_n = U^*((a_n)_n), \quad \forall (a_n)_n \in \ell^2(A).$$

□

The preceding theorem concludes our description of various classes of frames in terms of corresponding adjointable surjections (i.e., synthesis operators). The most important statements of this section and their mutual relations are shown in the diagram below.



*Remark 3.20.* Observe the bottom row of the above diagram: Theorem 3.12 and Corollary 3.14 establish a correspondence of strict frames for  $M(X)$  with adjointable surjections from  $M(\ell^2(A))$  to  $M(X)$ . On the other hand, since  $M(A)$  is a unital  $C^*$ -algebra, frames for  $M(X)$  correspond, by Proposition 3.1, to adjointable surjections from  $\ell^2(M(A))$  to  $M(X)$ .

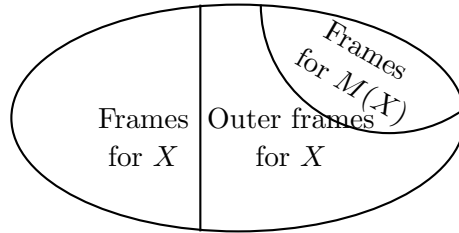
It is clear from the definition of a strict frame that the class of strict frames for  $M(X)$  contains the class of frames for  $M(X)$ . This reflects the fact that, in general,  $M(\ell^2(\mathbf{A}))$  is larger than  $\ell^2(M(\mathbf{A}))$ .

As an example of a strict frame which is not a frame for  $M(X)$  take again a strict Parseval frame  $(s_n)_n$  for  $M(\mathbb{K}(H)) = \mathbb{B}(H)$  from Example 3.5. To see that  $(s_n)_n$  is not a frame for  $\mathbb{B}(H)$ , suppose the opposite. Then we would have  $b = \sum_{n=1}^{\infty} s_n \langle s_n, b \rangle = \sum_{n=1}^{\infty} p_n b$  with the norm convergence for all  $b \in \mathbb{B}(H)$ , which is obviously impossible.

For reader's convenience we include a short overview of the preceding considerations concerning various classes of frames and their interrelations.

Let  $X$  be a countably generated Hilbert  $\mathbf{A}$ -module.

- If  $X = M(X)$ , i.e., if  $X$  is strictly complete (e.g., when  $\mathbf{A}$  is unital or when  $X$  is AFG) then there are no outer frames for  $X$  (Remark 3.4), strict frames coincide with frames (Theorem 3.18) and each surjection in  $\mathbb{B}(\ell^2(\mathbf{A}), X)$  is the synthesis operator of some frame for  $X$  (Theorem 3.19).
- If  $X \neq M(X)$  then, in particular,  $\mathbf{A}$  is non-unital and  $X$  is not AFG. The class of all strict frames for  $M(X)$  consists of two disjoint parts which are the classes of all frames for  $X$  and all outer frames for  $X$  (see the diagram below). Moreover, if the multiplier module  $M(X)$  is countably generated itself, then the class of outer frames for  $X$  contains as a subset all frames for  $M(X)$ . Each surjection in  $\mathbb{B}(\ell^2(\mathbf{A}), X)$  is the synthesis operator of some either frame or outer frame for  $X$  (Theorem 3.19).



Strict frames for  $M(X)$  in the case  $X \neq M(X)$ .

Finally, we include for future reference an easy consequence of the preceding results concerning Bessel sequences. The following corollary, together with Theorem 2.1 and Corollary 3.7, establishes a bijective correspondence between Bessel and outer Bessel sequences in a Hilbert  $C^*$ -module  $X$  and adjointable operators from  $\ell^2(\mathbf{A})$  to  $X$ .

**Corollary 3.21.** *Let  $X$  be a Hilbert  $A$ -module and let  $T \in \mathbb{B}(\ell^2(A), X)$ . Then there exists a unique Bessel sequence or outer Bessel sequence  $(x_n)_n$  in  $X$  whose synthesis operator coincides with  $T$ .*

*Proof.* If  $A$  is unital, put  $Te^{(n)} = x_n$ ,  $n \in \mathbb{N}$ . Then, obviously,  $T^*x = (\langle x_n, x \rangle)_n$  for all  $x \in X$ , and, since  $T^*$  is bounded,  $(x_n)_n$  is a Bessel sequence whose synthesis operator coincides with  $T$ .

If  $A$  is non-unital, put  $T_M e^{(n)} = x_n$ ,  $n \in \mathbb{N}$ . Again, it follows that  $T^*x = (\langle x_n, x \rangle)_n$  for all  $x \in X$ . Moreover,  $\sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle$  converges in norm for all  $x \in X$ . From this we conclude that  $(x_n)_n$  is a Bessel sequence, which turns out to be outer if at least one  $x_n$  belongs to  $M(X) \setminus X$ .  $\square$

We end the section with some comments on finite frames vs. adjointable surjections from  $A^N$ ,  $N \in \mathbb{N}$ , to the ambient Hilbert  $A$ -module  $X$ .

Let us first extend Definition 3.3 to finite sequences: if  $X$  is a Hilbert  $A$ -module, we say that a finite sequence  $(v_n)_{n=1}^N$ ,  $N \in \mathbb{N}$ , in  $M(X)$  is an outer frame for  $X$  if  $v_n \in M(X) \setminus X$  for at least one  $n \in \{1, \dots, N\}$ , and if there exist positive constants  $A$  and  $B$  such that

$$A\langle x, x \rangle \leq \sum_{n=1}^N \langle x, v_n \rangle \langle v_n, x \rangle \leq B\langle x, x \rangle, \quad \forall x \in X.$$

The analysis operator  $U$  is defined as

$$U : X \rightarrow A^N, \quad Ux = (\langle v_n, x \rangle)_{n=1}^N.$$

Its adjoint, the synthesis operator  $U^*$  is given by

$$U^*(a_1, \dots, a_N) = \sum_{n=1}^N v_n a_n.$$

**Proposition 3.22.** *Let  $X$  be a Hilbert  $A$ -module.*

- (a) *If there exists a finite frame for  $X$ , then  $X$  is AFG and, in particular, there are no outer frames for  $X$ .*
- (b) *If there exists a finite outer frame for  $X$  then  $M(X) \neq X$ ,  $X$  is not AFG, and  $A$  is a non-unital  $C^*$ -algebra. Moreover, then there are no finite frames for  $X$  and each finite outer frame for  $X$  is a frame for  $M(X)$ .*

*Proof.* To prove (a), we only need to recall that, by Remark 3.2(a),  $M(X) = X$  when  $X$  is AFG.

Similarly, if there exists an outer frame for  $X$  then, by definition,  $M(X) \neq X$  and again Remark 3.2(a) implies that then  $X$  is not AFG and  $A$  is non-unital. To prove the last statement in (b) we can argue as follows.

First, observe that  $M(A^N) = M(A)^N$ . Suppose that  $(v_n)_{n=1}^N$  is an outer frame for  $X$ , consider the analysis operator  $U \in \mathbb{B}(X, A^N)$  and its extension  $U_M \in \mathbb{B}(M(X), M(A)^N)$ . By Lemma 3.17,  $(U_M)^* \in \mathbb{B}(M(A)^N, M(X))$  is a

surjection. Let  $w_n = (U_M)^* e^{(n)}$ ,  $n = 1, \dots, N$ . As in the proof of Proposition 3.1, we easily conclude that  $(w_n)_{n=1}^N$  is a frame for  $M(X)$ . We now claim that  $w_n = v_n$  for all  $n = 1, \dots, N$ . To see this, take arbitrary  $n \in \{1, \dots, N\}$  and  $a \in A$ . Then

$$(U_M)^* a^{(n)} = (U_M)^* (e^{(n)} a) = \left( (U_M)^* e^{(n)} \right) a = w_n a,$$

and

$$(U_M)^* a^{(n)} = (U^*)_M a^{(n)} = U^* a^{(n)} = v_n a.$$

Thus,  $w_n a = v_n a$  and, since  $a$  was arbitrary, this is enough to conclude  $w_n = v_n$ .  $\square$

**Proposition 3.23.** *Let  $X$  be a Hilbert  $A$ -module and  $T \in \mathbb{B}(A^N, X)$ ,  $N \in \mathbb{N}$ , a surjection. Then there exists a unique frame or outer frame  $(x_n)_{n=1}^N$  for  $X$  whose synthesis operator coincides with  $T$ .*

*Proof.* If  $A$  is unital let  $x_n = T e^{(n)}$ ,  $n = 1, \dots, N$ . Then, clearly,  $(x_n)_{n=1}^N$  is a frame for  $X$  whose synthesis operator is  $T$ .

Suppose now that  $A$  is non-unital. Consider  $T_M \in \mathbb{B}(M(A)^N, M(X))$  and put  $x_n = T_M e^{(n)}$ ,  $n = 1, \dots, N$ . Since, by Lemma 3.17,  $T_M$  is a surjection,  $(x_n)_{n=1}^N$  is a frame for  $M(X)$  by [2, Theorem 2.5]. There are now two possibilities: either each  $x_n$  belongs to  $X$ , or  $x_n \in M(X) \setminus X$  for at least one  $n$ .

Assume first  $x_n \in X$  for all  $n = 1, \dots, N$ . By the reconstruction property it follows immediately that  $M(X) \subseteq X$ ; thus, in fact  $(x_n)_{n=1}^N$  is a frame for  $X$  and, in particular,  $X$  is AFG.

In the remaining possibility, if there exists  $n$  such that  $x_n \in M(X) \setminus X$ ,  $(x_n)_{n=1}^N$  is an outer frame for  $X$  and, in particular, Proposition 3.22(b) applies.

In both cases the corresponding synthesis operator coincides with  $T$ .  $\square$

Note that the situation described in Proposition 3.22(b) means that  $X$ , although not algebraically generated by finitely many elements, admits finite outer frames. It is not difficult to find examples of such Hilbert  $C^*$ -modules. In fact, every non-unital  $C^*$ -algebra  $A$  serves as a simple example of this kind.

To see this, take any non-unital  $C^*$ -algebra  $A$  and regard it as a Hilbert  $A$ -module over itself. Since  $\mathbb{K}(A) = A$  is non-unital,  $A$  is not AFG as a Hilbert  $C^*$ -module, and therefore there are no finite frames for  $A$ . On the other hand, here the multiplier algebra  $M(A)$  plays the role of the multiplier module, so the unit element  $e \in M(A)$  serves as a frame for  $M(A)$  and an outer frame for  $A$ . This is, indeed, obvious from the equality  $a = e \langle e, a \rangle$  that is trivially satisfied for all  $a \in A$ .

Having obtained necessary results on outer frames we are now ready for a detailed study of various questions (such as dual frames, perturbations, tight approximations, etc) that are prominent for the frame theory. This is the



purpose of the second part of the paper. In our study we shall be interested primarily in countably generated Hilbert  $C^*$ -modules and their frames, but as we shall see, outer frames will naturally appear into the picture. Hence the results that follow will be concerned with both frames and outer frames.

It should be noted that some of that results are valid even for Hilbert  $C^*$ -modules that are not countably generated (in the usual sense), but which are countably generated in  $M(X)$ .

#### 4. DUAL FRAMES

Dual frames for Hilbert  $C^*$ -modules were introduced and discussed in [10], Section 6, where the existence of canonical and alternate dual frames was established, and some of their fundamental properties were proven.

Suppose that  $(x_n)_n$  is a frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . Then we know that the sequence  $(y_n)_n = ((U^*U)^{-1}x_n)_n$  is also a frame for  $X$  which satisfies  $x = \sum_{n=1}^{\infty} y_n \langle x_n, x \rangle$  for all  $x \in X$ . The frame  $(y_n)_n$  is called the canonical dual of  $(x_n)_n$ . If we denote its analysis operator by  $V$  then the preceding equality can be rewritten in the form  $V^*U = I$ . It is now natural to try to describe all frames  $(z_n)_n$  for  $X$  that are dual to  $(x_n)_n$  in the sense that the equality  $x = \sum_{n=1}^{\infty} z_n \langle x_n, x \rangle$  is satisfied for each  $x$  in  $X$ . If  $W$  denotes the analysis operator of  $(z_n)_n$ , this simply means  $W^*U = I$ .

Hence, the problem of finding dual frames of  $(x_n)_n$  is closely related to the problem of finding solutions of the equation  $TU = I$  with  $T \in \mathbb{B}(\ell^2(A), X)$ . Obviously, each  $T$  such that  $TU = I$  is surjective. When  $A$  is unital, we know by Proposition 3.1 that such  $T$  is the synthesis operator of some frame for  $X$ , and one immediately concludes (see Lemma 4.3 below) that the obtained frame is dual to  $(x_n)_n$ .

The non-unital case is more complicated because among solutions of  $TU = I$  there might be adjointable surjections which are not synthesis operators of frames for  $X$ . However, by Theorem 3.19, such surjections are synthesis operators of outer frames for  $X$ , and it will turn out that each outer frame  $(y_n)_n$  obtained in that way also satisfies  $x = \sum_{n=1}^{\infty} y_n \langle x_n, x \rangle$  for all  $x \in X$ . (Indeed, outer duals do exist; see Examples 4.4 and 4.5 below.)

Therefore, by solving the equation  $TU = I$  in  $\mathbb{B}(\ell^2(A), X)$  we shall get synthesis operators of both frames and outer frames for  $X$  dual to a given frame.

This suggests a need for a unified treatment of dual frames, without a priori distinguishing between frames and dual frames.

Before embarking into our study, let us point out that here we shall restrict ourselves to (outer) frames and (outer) Bessel sequences. That is, we are not going to discuss general sequences that behave like duals to a given frame.

Recall that even in a Hilbert space in some situations there are sequences that are not even Bessel, but which are dual to a given frame.

Throughout this section all our statements are concerned only with infinite frames and outer frames. A short remark about the finite case is included at the end of the section.

Further, if  $Y$  is a complementable closed Hilbert  $C^*$ -submodule of  $X$  (i.e.,  $X = Y \oplus Y^\perp$ ), we denote by  $P_Y \in \mathbb{B}(X)$  the orthogonal projection to  $Y$ . Recall that a closed Hilbert  $C^*$ -submodule  $Y$  of  $X$  is complementable in  $X$  if and only if  $Y$  is the range of an adjointable operator (see e.g. [23, Corollary 15.3.9]).

Let us start with a definition.

*Definition 4.1.* Let  $X$  be a Hilbert  $A$ -module and  $(x_n)_n$  a frame or an outer frame for  $X$ . A frame or an outer frame  $(y_n)_n$  for  $X$  is said to be a *dual to*  $(x_n)_n$  if

$$(30) \quad \sum_{n=1}^{\infty} y_n \langle x_n, x \rangle = x, \quad \forall x \in X.$$

*Remark 4.2.* Let  $(x_n)_n$  and  $(y_n)_n$  be as in the above definition; denote by  $U$  and  $V$  the analysis operators of  $(x_n)_n$  and  $(y_n)_n$ , respectively. Then, obviously, (30) can be rewritten as

$$(31) \quad V^*U = I,$$

which is equivalent to

$$(32) \quad U^*V = I,$$

or

$$(33) \quad \sum_{n=1}^{\infty} x_n \langle y_n, x \rangle = x, \quad \forall x \in X.$$

Hence, as long as we work with frames and outer frames (*not mere sequences*), equalities (30) to (33) are mutually equivalent, duality is a symmetric relation, and we can say that  $(x_n)_n$  and  $(y_n)_n$  are dual to each other.

Moreover, our first lemma will show, generalizing [13, Proposition 3.8], that the same is true for Bessel sequences and outer Bessel sequences.

**Lemma 4.3.** *Let  $X$  be a Hilbert  $A$ -module. If  $(x_n)_n$  and  $(y_n)_n$  are Bessel or outer Bessel sequences in  $X$  with the analysis operators  $U$  and  $V$ , respectively, satisfying at least one of equalities (30) to (33), then  $(x_n)_n$  and  $(y_n)_n$  are frames or outer frames for  $X$ , they satisfy all equalities (30) to (33), and are dual to each other.*

*Proof.* First note, by Theorem 2.1 and Corollaries 3.7 and 3.21, that Bessel sequences and outer Bessel sequences correspond to adjointable operators from  $X$  to  $\ell^2(A)$ , so that not only all four above equalities make sense, but also each of them implies the remaining three.

So, suppose that equalities (30) to (33) hold. From (31) we get that  $V^*$  is a surjection. By Theorem 3.19,  $(y_n)_n$  is a frame or an outer frame for  $X$ . By invoking (32), the same argument applies to  $(x_n)_n$ .  $\square$

If  $X$  is a strictly complete Hilbert  $C^*$ -module (i.e., if  $M(X) = X$ ) our discussion on duality reduces to frames since then there are no outer frames. When  $M(X) \neq X$ , the situation is more complicated. If  $(x_n)_n$  is a frame for  $X$ , its canonical dual is also a frame. On the other hand, if  $(x_n)_n$  is an outer frame for  $X$ , its canonical dual is also outer. Both statements follow from the fact that the  $(U^*U)^{-1}$  acts bijectively on  $X$ . In general, a frame  $(x_n)_n$  for  $X$  may have outer dual frames. Indeed, in two examples that follow we demonstrate that:

- each countably generated Hilbert  $C^*$ -module  $X$  such that  $M(X) \neq X$  has a frame which possesses an outer dual frame,
- there exists a frame for a countably generated Hilbert  $C^*$ -module  $X$  possessing an outer dual frame whose all elements are in  $M(X) \setminus X$ .

*Example 4.4.* Let  $X$  be a countably generated Hilbert  $A$ -module that is not strictly complete, i.e.,  $X \neq M(X)$ . Take any Parseval frame  $(x_n)_{n=1}^\infty$  for  $X$ . Let  $x_0 \in X$  and  $y_0 \in M(X) \setminus X$  be such that  $\theta_{y_0, x_0} = 0$  (for example, we can take  $x_0 = 0$  and arbitrary  $y_0 \in M(X) \setminus X$ ). Let  $y_n = x_n$  for  $n \in \mathbb{N}$ .

Then  $(x_n)_{n=0}^\infty$  is a frame for  $X$ ,  $(y_n)_{n=0}^\infty$  is an outer frame for  $X$ , and they are dual to each other since

$$\sum_{n=0}^{\infty} y_n \langle x_n, x \rangle = y_0 \langle x_0, x \rangle + \sum_{n=1}^{\infty} x_n \langle x_n, x \rangle = x, \quad \forall x \in X.$$

*Example 4.5.* Let  $(\epsilon_n)_n$  be an orthonormal basis of a separable Hilbert space  $H$ . Denote by  $(\cdot | \cdot)$  the inner product in  $H$ . Consider  $\mathbb{K} = \mathbb{K}(H)$  as a Hilbert  $\mathbb{K}$ -module in the standard way.

For  $i, j \in \mathbb{N}$  let  $e_{i,j} \in \mathbb{B}(H)$  be the 1-dimensional partial isometry defined by  $e_{i,j}(\xi) = (\xi | \epsilon_j) \epsilon_i$ ,  $\xi \in H$ . In particular, for each  $n \in \mathbb{N}$ ,  $e_{n,n}$  is the orthogonal projection to  $\text{span}\{\epsilon_n\}$ . One easily verifies that  $e_{i,j} e_{k,l} = \delta_{j,k} e_{i,l}$  and  $e_{i,j}^* = e_{j,i}$  for all  $i, j, k, l \in \mathbb{N}$ .

Since  $(\sum_{n=1}^N \theta_{e_{n,1}, e_{n,1}})_N = (\sum_{n=1}^N e_{n,1} e_{n,1}^*)_N = (\sum_{n=1}^N e_{n,n})_N$  is an approximate unit for  $\mathbb{K}$ , Proposition 2.3 implies that  $(e_{n,1})_n$  is a Parseval frame for  $\mathbb{K}$ .

Let  $(H_n)_n$  be a sequence of closed infinite dimensional subspaces of  $H$  such that  $\bigoplus_{n=1}^\infty H_n = H$ . For each  $n \in \mathbb{N}$  consider a partial isometry  $t_n \in \mathbb{B}(H)$  such that  $\text{N}(t_n) = \text{span}\{\epsilon_1\}$  and  $\text{R}(t_n) = H_n$ . Thus,  $t_n t_n^*$  is the orthogonal projection to  $H_n$  for all  $n \in \mathbb{N}$ . By construction,  $t_n e_{1,n} = 0$  for all  $n \in \mathbb{N}$ . As in Example 1.7 one verifies that the series  $\sum_{n=1}^\infty t_n t_n^* a$  converges in norm to  $a$ , for each  $a$  in  $\mathbb{K}$ .

Let  $y_n = e_{n,1} + t_n$ ,  $n \in \mathbb{N}$ . Then for all  $a \in \mathbb{K}$  we have

$$\sum_{n=1}^{\infty} y_n \langle y_n, a \rangle = \sum_{n=1}^{\infty} (e_{n,1} + t_n)(e_{1,n} + t_n^*)a = \sum_{n=1}^{\infty} e_{n,n}a + \sum_{n=1}^{\infty} t_n t_n^* a = 2a,$$

and since  $y_n \notin \mathbb{K}$  for every  $n \in \mathbb{N}$ , we conclude that  $(y_n)_n$  is an outer 2-tight frame for  $\mathbb{K}$ .

Finally,  $(y_n)_n$  and  $(e_{n,1})_n$  are dual to each other, since for all  $a \in \mathbb{K}$

$$\sum_{n=1}^{\infty} y_n \langle e_{n,1}, a \rangle = \sum_{n=1}^{\infty} (e_{n,1} + t_n)e_{1,n}a = \sum_{n=1}^{\infty} e_{n,n}a = a.$$

We now state our first result which describes all frames and outer frames that are dual to a given one.

**Theorem 4.6.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . An operator  $V \in \mathbb{B}(X, \ell^2(A))$  is the analysis operator of a frame or an outer frame dual to  $(x_n)_n$  if and only if  $V$  is of the form*

$$(34) \quad V = U(U^*U)^{-1} + (I - U(U^*U)^{-1}U^*)L$$

for some  $L \in \mathbb{B}(X, \ell^2(A))$ .

*Proof.* If  $(y_n)_n$  is a frame or an outer frame dual to  $(x_n)_n$ , i.e., if  $U^*V = I$ , then (34) is fulfilled if we choose  $L = V$ .

Conversely, if  $V$  is as in (34), then a straightforward verification shows that  $U^*V = I$  and Lemma 4.3 applies.  $\square$

*Remark 4.7.* Suppose we are given a frame or an outer frame  $(x_n)_n$  for a Hilbert  $C^*$ -module  $X$ . Denote the corresponding analysis operator by  $U$ .

Let  $P = I - U(U^*U)^{-1}U^* \in \mathbb{B}(\ell^2(A))$ . It is easy to verify that  $P = P^* = P^2$ , and  $P((a_n)_n) = (a_n)_n$  if and only if  $(a_n)_n \in N(U^*)$ . Since  $U$  has a closed range,  $R(U)$  is complementable in  $\ell^2(A)$  and  $N(U^*) = R(U)^\perp$ . Therefore,

$$(35) \quad I - U(U^*U)^{-1}U^* = P_{R(U)^\perp}.$$

Observe that each  $V$  as in (34) consists of two terms. The first one is just the analysis operator of the canonical dual of  $(x_n)_n$ . The second term comes from an arbitrary adjointable operator  $L : X \rightarrow \ell^2(A)$  compressed to the submodule  $R(U)^\perp = N(U^*)$  which is a part of  $\ell^2(A)$  of no relevance as far as the right inverse of  $U^*$  is concerned.

**Corollary 4.8.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . If  $(y_n)_n$  is a frame or an outer frame dual to  $(x_n)_n$  then*

$$(36) \quad \sum_{n=1}^{\infty} \langle x, (U^*U)^{-1}x_n \rangle \langle (U^*U)^{-1}x_n, x \rangle \leq \sum_{n=1}^{\infty} \langle x, y_n \rangle \langle y_n, x \rangle, \quad \forall x \in X.$$

*Proof.* Let  $V$  be the analysis operator for  $(y_n)_n$ . For every  $x \in X$  the left hand side of (36) is equal to

$$\begin{aligned} \sum_{n=1}^{\infty} \langle x, (U^*U)^{-1}x_n \rangle \langle (U^*U)^{-1}x_n, x \rangle &= \sum_{n=1}^{\infty} \langle (U^*U)^{-1}x, x_n \rangle \langle x_n, (U^*U)^{-1}x \rangle \\ &= \langle U(U^*U)^{-1}x, U(U^*U)^{-1}x \rangle \\ &= \langle (U^*U)^{-1}x, x \rangle, \end{aligned}$$

while the right hand side is  $\langle Vx, Vx \rangle = \langle V^*Vx, x \rangle$ . Therefore, (36) reads as

$$(37) \quad (U^*U)^{-1} \leq V^*V.$$

By Theorem 4.6 and Remark 4.7, there is  $L \in \mathbb{B}(X, \ell^2(A))$  such that  $V = U(U^*U)^{-1} + P_{R(U)^\perp}L$ . Then a straightforward calculation shows that  $V^*V = (U^*U)^{-1} + L^*P_{R(U)^\perp}L$ , which obviously implies  $V^*V \geq (U^*U)^{-1}$ .  $\square$

We note that the above corollary sharpens Proposition 6.5 from [10]. It shows that the frame coefficients of the canonical dual retain the minimality property even when outer frames are included into consideration.

Theorem 4.6 enables us also to describe those frames and outer frames that possess Parseval duals.

**Corollary 4.9.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . Then  $(x_n)_n$  admits a Parseval dual or an outer Parseval dual  $(y_n)_n$  if and only if there is  $T \in \mathbb{B}(X, \ell^2(A))$  such that  $U^*U - I = T^*P_{R(U)^\perp}T$ .*

*Proof.* Suppose  $(x_n)_n$  admits a Parseval dual frame or outer frame  $(y_n)_n$ . If  $V$  is the analysis operator of  $(y_n)_n$  then, by Theorem 4.6, there exists  $L \in \mathbb{B}(X, \ell^2(A))$  such that  $V = U(U^*U)^{-1} + P_{R(U)^\perp}L$ . Then  $I = V^*V = (U^*U)^{-1} + L^*P_{R(U)^\perp}L$ , so, denoting  $T = L(U^*U)^{\frac{1}{2}}$ , we get

$$\begin{aligned} U^*U - I &= (U^*U)^{\frac{1}{2}}(I - (U^*U)^{-1})(U^*U)^{\frac{1}{2}} \\ &= (U^*U)^{\frac{1}{2}}L^*P_{R(U)^\perp}L(U^*U)^{\frac{1}{2}} \\ &= T^*P_{R(U)^\perp}T. \end{aligned}$$

Conversely, suppose  $U^*U - I = T^*P_{R(U)^\perp}T$  for some  $T \in \mathbb{B}(X, \ell^2(A))$ . Let  $V = U(U^*U)^{-1} + P_{R(U)^\perp}T(U^*U)^{-\frac{1}{2}}$ . By Theorem 4.6,  $V$  is the analysis operator of some frame or outer frame  $(y_n)_n$  for  $X$  which is dual to  $(x_n)_n$ . Since

$$\begin{aligned} V^*V &= (U^*U)^{-1} + (U^*U)^{-\frac{1}{2}}T^*P_{R(U)^\perp}T(U^*U)^{-\frac{1}{2}} \\ &= (U^*U)^{-\frac{1}{2}}(I + T^*P_{R(U)^\perp}T)(U^*U)^{-\frac{1}{2}} \\ &= (U^*U)^{-\frac{1}{2}}U^*U(U^*U)^{-\frac{1}{2}} = I, \end{aligned}$$

$(y_n)_n$  is a Parseval frame or an outer Parseval frame for  $X$ .  $\square$

*Remark 4.10.* Hilbert space frames that possess Parseval duals are described in [12] and [1], see also [4]. It turns out that a frame  $(x_n)_n$  for a Hilbert space possesses a Parseval dual if and only if  $A \geq 1$  and  $\dim \mathcal{R}(U^*U - I) \leq \dim \mathcal{R}(U)^\perp$ . (Here, as usual,  $A$  denotes a lower frame bound and  $U$  is the analysis operator.) Note that the later condition means that  $\mathcal{R}(U^*U - I)$  can be isometrically embedded into  $\mathcal{R}(U)^\perp$ . We note that these conditions are implied by Corollary 4.9.

Indeed, if  $U^*U - I = T^*P_{\mathcal{R}(U)^\perp}T$  then, obviously,  $U^*U - I \geq 0$  which implies  $A \geq 1$ . On the other hand, the equality  $U^*U - I = T^*P_{\mathcal{R}(U)^\perp}T$  can be rewritten as  $(U^*U - I)^{\frac{1}{2}}(U^*U - I)^{\frac{1}{2}} = T^*P_{\mathcal{R}(U)^\perp}T$  which gives us

$$\left\langle (U^*U - I)^{\frac{1}{2}}x, (U^*U - I)^{\frac{1}{2}}x \right\rangle = \left\langle P_{\mathcal{R}(U)^\perp}Tx, P_{\mathcal{R}(U)^\perp}Tx \right\rangle, \quad \forall x \in X.$$

Using this equality, we can define a map  $\phi : \mathcal{R}((U^*U - I)^{\frac{1}{2}}) \rightarrow \mathcal{R}(U)^\perp$  by  $\phi((U^*U - I)^{\frac{1}{2}}x) = P_{\mathcal{R}(U)^\perp}Tx$ . Clearly,  $\phi$  is a well-defined isometry. Since, obviously,  $\mathcal{R}(U^*U - I) \subseteq \mathcal{R}((U^*U - I)^{\frac{1}{2}})$ , we conclude that the above map  $\phi$  provides an isometrical embedding of  $\mathcal{R}(U^*U - I)$  into  $\mathcal{R}(U)^\perp$ .

Next we provide another description of all frames and outer frames that are dual to a given one. We shall use [4] as a blueprint.

**Proposition 4.11.** *Let  $X$  and  $Y$  be Hilbert  $A$ -modules. Let  $U \in \mathbb{B}(X, Y)$  and  $T \in \mathbb{B}(Y, X)$  be such that  $TU = I$ . Then*

- (a)  $\mathcal{N}(T) = \mathcal{R}(I - UT) = (I - UT)(\mathcal{N}(U^*))$
- (b)  $Y = \mathcal{R}(U) \dot{+} \mathcal{N}(T)$  (a direct sum),
- (c)  $UT \in \mathbb{B}(Y)$  is the oblique projection to  $\mathcal{R}(U)$  along  $\mathcal{N}(T)$ .

*Proof.* (a) From  $T(I - UT) = 0$  we have  $\mathcal{R}(I - UT) \subseteq \mathcal{N}(T)$ . Conversely,  $y \in \mathcal{N}(T) \Rightarrow (I - UT)y = y \Rightarrow y \in \mathcal{R}(I - UT)$ . This gives the first equality.

To prove  $\mathcal{R}(I - UT) \subseteq (I - UT)(\mathcal{N}(U^*))$  (the opposite inclusion is obvious), first observe that our assumption  $TU = I$  implies that  $U$  is bounded from below. Hence  $\mathcal{R}(U)$  is a closed submodule of  $Y$  and  $Y = \mathcal{R}(U) \oplus \mathcal{N}(U^*)$ . Let us now take arbitrary  $(I - UT)y \in \mathcal{R}(I - UT)$ ,  $y \in Y$ . Then  $y = Ux + z$  for some  $x \in X$  and  $z \in \mathcal{N}(U^*)$ , so we have

$$(I - UT)y = (I - UT)Ux + (I - UT)z = (I - UT)z \in (I - UT)(\mathcal{N}(U^*)).$$

(b) Let  $y \in \mathcal{R}(U) \cap \mathcal{N}(T)$ . Then  $y = Ux$  for some  $x \in X$  and  $Ty = 0$ . Putting this together we get  $TUx = 0$ ; thus, by assumption,  $x = 0$ . Hence,  $y = 0$  and this shows that the intersection  $\mathcal{R}(U) \cap \mathcal{N}(T)$  is trivial.

Let us now take arbitrary  $y \in Y$  and write it, as in the preceding paragraph, in the form  $y = Ux + z$  with  $x \in X$  and  $z \in \mathcal{N}(U^*)$ . Then we have (again as before)  $(I - UT)y = (I - UT)z$ . This can be rewritten as

$$(38) \quad y = UTy + (I - UT)z.$$

Since  $UTy \in \mathcal{R}(U)$  and  $(I - UT)z \in (I - UT)(\mathcal{N}(U^*)) \stackrel{(a)}{=} \mathcal{N}(T)$ , the proof is completed.

(c) Evidently,  $UT \in \mathbb{B}(Y)$  satisfies  $UTUx = Ux$  for all  $Ux \in R(U)$ , and  $UTy = 0$  for all  $y \in N(T)$ .  $\square$

**Proposition 4.12.** *Let  $(x_n)_n$  and  $(y_n)_n$  be frames or outer frames for a Hilbert  $A$ -module  $X$  that are dual to each other. Denote by  $U$  and  $V$  the corresponding analysis operators. Then*

- (a)  $\ell^2(A) = R(U) \dot{+} N(V^*)$ ,
- (b)  $UV^*$  is the oblique projection to  $R(U)$  along  $N(V^*)$ ,
- (c)  $\ell^2(A) = R(V) \dot{+} N(U^*)$ ,
- (d)  $VU^*$  is the oblique projection to  $R(V)$  along  $N(U^*)$ .

*Proof.* (a) and (b) follow from the preceding proposition and the equality  $V^*U = I$ , while (c) and (d) are obtained in the same way using the equality  $U^*V = I$ .  $\square$

*Remark 4.13.* Consider a frame or an outer frame  $(x_n)_n$  for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . Let  $(y_n)_n$  be a frame or an outer frame dual to  $(x_n)_n$ ; let  $V$  denotes the corresponding analysis operator. Then, by the preceding proposition,  $UV^*$  is an oblique projection to  $R(U)$ . In the special case when  $(y_n)_n$  is the canonical dual of  $(x_n)_n$  we have  $V = U(U^*U)^{-1}$  and  $UV^* = U(U^*U)^{-1}U^*$  which is by Remark 4.7 the orthogonal projection to  $R(U)$ . So, in the light of the preceding proposition, this orthogonality is the exclusive property of the canonical dual among all frames and outer frames that are dual to  $(x_n)_n$ .

The following theorem is a result similar to Theorem 4.6. It provides another characterization of analysis operators of dual frames and outer frames.

**Theorem 4.14.** *Let  $(x_n)_n$  be a frame or an outer frame for Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . An operator  $V \in \mathbb{B}(X, \ell^2(A))$  is the analysis operator of a frame or an outer frame dual to  $(x_n)_n$  if and only if  $V$  is of the form*

$$(39) \quad V = F^*U(U^*U)^{-1},$$

where  $F \in \mathbb{B}(\ell^2(A))$  is an oblique projection to  $R(U)$  along some closed direct complement of  $R(U)$  in  $\ell^2(A)$ .

*Proof.* If  $(y_n)_n$  is a frame or an outer frame dual to  $(x_n)_n$  then its analysis operator  $V \in \mathbb{B}(X, \ell^2(A))$  satisfies  $V^*U = I$  so, by Proposition 4.12,  $\ell^2(A) = R(U) \dot{+} N(V^*)$  and  $UV^*$  is the oblique projection to  $R(U)$  along  $N(V^*)$ . Let  $F = UV^*$ . Then  $F^*U(U^*U)^{-1} = VU^*U(U^*U)^{-1} = V$ .

To prove the converse, take  $V$  as in (39) and observe that  $FU = U$ . Then we have  $V^*U = (U^*U)^{-1}U^*FU = I$ .  $\square$

We conclude this section with a discussion about frames and outer frames that have a unique dual.

**Theorem 4.15.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . Consider the following conditions:*

- (a)  $N(U^*) = \{0\}$ .
- (b)  $R(U) = \ell^2(A)$ .
- (c) *The canonical dual is the only dual (including both frames and outer frames) of  $(x_n)_n$ .*

*Then (a) and (b) are mutually equivalent and imply (c). If  $X$  is full, (c) is equivalent to (a) and (b).*

*Proof.* Since  $\ell^2(A) = R(U) \oplus N(U^*)$ , (a) and (b) are equivalent. Also, (b) together with Theorem 4.14 immediately implies (c).

To prove the last statement, suppose that  $X$  is full and that (c) is satisfied. Recall from Theorem 4.6 that each adjointable operator  $L : X \rightarrow \ell^2(A)$  gives rise to a frame or an outer frame for  $X$  that is dual to  $(x_n)_n$  and whose analysis operator is given by

$$V = U(U^*U)^{-1} + (I - U(U^*U)^{-1}U^*)L.$$

By (c), we now have  $(I - U(U^*U)^{-1}U^*)L = 0$  for all  $L \in \mathbb{B}(X, \ell^2(A))$ , or equivalently,  $L = U(U^*U)^{-1}U^*L$  for all  $L \in \mathbb{B}(X, \ell^2(A))$ . Recall from Remark 4.7 that  $U(U^*U)^{-1}U^*$  is the orthogonal projection to  $R(U)$ . Hence, the above conclusion means that each operator  $L \in \mathbb{B}(X, \ell^2(A))$  takes values in  $R(U)$ .

Let us now take arbitrary  $x \in X$  and  $j \in \mathbb{N}$ . Define  $L_{x,j} : X \rightarrow \ell^2(A)$  by  $L_{x,j}(y) = (0, \dots, 0, \langle x, y \rangle, 0, \dots)$  (with  $\langle x, y \rangle$  on  $j$ -th position). Obviously,  $L_{x,j}$  is an adjointable operator whose adjoint is given by  $L_{x,j}^*((a_n)_n) = xa_j$ . By the preceding conclusion, all  $L_{x,j}$  take values in  $R(U)$ . Since  $X$  is by our assumption full, this immediately implies that  $c_{00}(A) \subseteq R(U)$ . Since  $R(U)$  is closed, this gives us  $\ell^2(A) \subseteq R(U)$  and hence  $R(U) = \ell^2(A)$ .  $\square$

Here we need to make a comment on Theorem 3.10 from [13]. Namely, that theorem states that all three above conditions are equivalent without assuming that the ambient Hilbert module  $X$  is full over  $A$ . However, there is a gap in the proof of Theorem 3.10 from [13] and this is the reason why we decided to include the preceding theorem in the present paper.

To show that the fullness assumption is really necessary in the proof of the implication (c)  $\Rightarrow$  (b) from Theorem 4.15, we provide the following example.

Let  $B$  be a unital  $C^*$ -algebra that is contained as a non-essential ideal in a unital  $C^*$ -algebra  $A$ . This means that  $B^\perp = \{a \in A : aB = \{0\}\} \neq \{0\}$ . Consider  $X = \ell^2(B)$  as a Hilbert  $C^*$ -module over  $A$ . Clearly,  $X$  is not full as a Hilbert  $A$ -module. Denote by  $e$  the unit element of  $B$ . Obviously, the sequence  $(e^{(n)})_n$  is a Parseval frame for  $X$ . One easily concludes that the corresponding analysis operator  $U : X \rightarrow \ell^2(A)$  acts as the inclusion; hence



$R(U) = \ell^2(\mathbf{B})$ . This means that  $U$  is not a surjection and that  $R(U)^\perp = N(U^*)$  is a non-trivial submodule of  $\ell^2(\mathbf{A})$ .

However,  $(e^{(n)})_n$  has a unique dual frame (in fact,  $(e^{(n)})_n$  is, being Parseval, self-dual). To prove this, recall that the analysis operator of each frame dual to  $(e^{(n)})_n$  (here there are no outer frame since  $\mathbf{A}$  is unital) is given by

$$V = U(U^*U)^{-1} + (I - U(U^*U)^{-1}U^*)L,$$

where  $L : X \rightarrow \ell^2(\mathbf{A})$  is an adjointable operator. We now observe that the Hewitt-Cohen factorization (Proposition 2.31 from [21]) forces each  $L$  to take values in  $R(U) = \ell^2(\mathbf{B})$ . Since, by Remark 4.7,  $I - U(U^*U)^{-1}U^*$  is the orthogonal projection to  $R(U)^\perp$ , we have  $(I - U(U^*U)^{-1}U^*)L = 0$  for each  $L \in \mathbb{B}(X, \ell^2(\mathbf{A}))$ . This together with the fact  $U^*U = I$  shows that the above equality reduces, for all  $L$ , to  $V = U$ . Hence, there is only one dual frame, namely  $(e^{(n)})_n$  itself.

Let us now state several consequences of Theorem 4.15.

**Corollary 4.16.** *A full Hilbert  $\mathbf{A}$ -module  $X$  which possesses a frame or an outer frame  $(x_n)_n$  with a unique dual is unitarily equivalent to  $\ell^2(\mathbf{A})$ .*

*Proof.* If  $(x_n)_n$  is a frame or an outer frame for  $X$  which has a unique dual, then its analysis operator  $U \in \mathbb{B}(X, \ell^2(\mathbf{A}))$  is invertible by Theorem 4.15, so the operator  $U(U^*U)^{-\frac{1}{2}} \in \mathbb{B}(X, \ell^2(\mathbf{A}))$  is unitary.  $\square$

**Corollary 4.17.** *Let  $X$  be a full Hilbert  $C^*$ -module over a non-unital  $C^*$ -algebra  $\mathbf{A}$ . Then every frame for  $X$  has at least two duals.*

*Proof.* Suppose there is a frame  $(x_n)_n$  for  $X$  with the unique dual. Let  $U \in \mathbb{B}(X, \ell^2(\mathbf{A}))$  be the corresponding analysis operator. By Theorem 4.15,  $U$  is a bijection.

Regarding  $X$  as a Hilbert  $\tilde{\mathbf{A}}$ -module, it is easy to verify that  $U$  can be regarded as an adjointable operator  $\tilde{U} \in \mathbb{B}(X, \ell^2(\tilde{\mathbf{A}}))$  given by  $\tilde{U}x = Ux$ ,  $x \in X$ . Then  $R(\tilde{U}) = R(U) = \ell^2(\mathbf{A})$ . Since  $\tilde{U}$  is bounded from below, its range  $R(\tilde{U})$  is closed in  $\ell^2(\tilde{\mathbf{A}})$ . So, being the range of an adjointable operator, a closed submodule  $\ell^2(\mathbf{A})$  of  $\ell^2(\tilde{\mathbf{A}})$  must be complementable in  $\ell^2(\tilde{\mathbf{A}})$ . But this is a contradiction since  $\ell^2(\mathbf{A})^\perp = \{0\}$ . (Namely, if  $(b_n)_n \in \ell^2(\tilde{\mathbf{A}})$  belongs to  $\ell^2(\mathbf{A})^\perp$ , then for each  $m$  it holds  $b_m a = \langle (b_n)_n, (a^{(m)})_n \rangle = 0$  for all  $a \in \mathbf{A}$ . Since  $\mathbf{A}$  is an essential ideal of  $\tilde{\mathbf{A}}$ , it follows that  $b_m = 0$  for all  $m$ .)  $\square$

*Remark 4.18.* Corollary 4.17 does not hold for outer frames. Indeed, if  $\mathbf{A}$  is a non-unital  $C^*$ -algebra and  $X = \ell^2(\mathbf{A})$ , then  $(e^{(n)})_n$  is an outer frame for  $X$  whose analysis operator  $U$  is the identity, so by Theorem 4.15,  $(e^{(n)})_n$  has a unique dual.

*Remark 4.19.* By Corollary 4.16 generalized Hilbert spaces  $\ell^2(\mathbf{A})$  with  $\mathbf{A}$   $\sigma$ -unital are, up to unitary equivalence, only countably generated Hilbert  $C^*$ -modules that possess frames or outer frames with unique duals. If  $\mathbf{A}$  is unital,  $(e^{(n)})_n$  is a Parseval frame for  $\ell^2(\mathbf{A})$  with this property. If  $\mathbf{A}$  is non-unital, Corollary 4.17 tells us that such frames in  $\ell^2(\mathbf{A})$  do not exist, so we only have outer frames with unique duals. As the example from the preceding remark shows,  $(e^{(n)})_n$  is such an outer frame.

For our last result of this section recall that each frame or outer frame  $(x_n)_n$  for a Hilbert  $C^*$ -module  $X$  possesses canonically associated Parseval frame or outer Parseval frame  $(y_n)_n$ . If  $U$  denotes the analysis operator of  $(x_n)_n$ ,  $y_n$ 's are given by  $y_n = (U_M^* U_M)^{-\frac{1}{2}} x_n$ ,  $n \in \mathbb{N}$ . Here we must work with the extended operator  $U_M$  if  $(x_n)_n$  is outer. If, on the other hand,  $x_n \in X$ , for all  $n \in \mathbb{N}$ , then the preceding equality reduces to  $y_n = (U^* U)^{-\frac{1}{2}} x_n \in X$ ,  $n \in \mathbb{N}$ . Observe that in both cases the analysis operator of  $(y_n)_n$  is given by  $U(U^* U)^{-\frac{1}{2}}$ .

In the following corollary we consider a Hilbert  $C^*$ -module over a unital  $C^*$ -algebra (because of Corollary 4.17), so there are no outer frames.

**Corollary 4.20.** *Let  $X$  be a full Hilbert  $C^*$ -module over a unital  $C^*$ -algebra  $A$ . Suppose there exists a frame  $(x_n)_n$  with a unique dual. Let  $U$  be the analysis operator for  $(x_n)_n$ . Then the following statements hold:*

- (a) *The Parseval frame  $(y_n)_n$  canonically associated with  $(x_n)_n$  has a unique dual and*

$$\langle y_n, y_m \rangle = \delta_{nm} e, \quad \forall m, n \in \mathbb{N}.$$

- (b)  *$\langle x_n, x_n \rangle$  is invertible for every  $n$ .*

- (c) *If  $\sum_{n=1}^{\infty} x_n a_n = 0$  for some  $a_n \in A$ ,  $n \in \mathbb{N}$ , then  $a_n = 0$  for all  $n \in \mathbb{N}$ .*

*Proof.* By Theorem 4.15,  $U$  is a bijection. Since  $\mathbf{A}$  is unital,  $x_n = U^*(e^{(n)})$  for every  $n \in \mathbb{N}$ . The analysis operator  $V = U(U^* U)^{-\frac{1}{2}}$  for  $(y_n)_n$  is an isometry and a bijection, hence unitary, so

$$\langle y_n, y_m \rangle = \langle V^*(e^{(n)}), V^*(e^{(m)}) \rangle = \langle (e^{(n)}), (e^{(m)}) \rangle = \delta_{nm} e, \quad \forall m, n \in \mathbb{N}.$$

Further,

$$e = \langle y_n, y_n \rangle = \langle (U^* U)^{-1} x_n, x_n \rangle \leq \| (U^* U)^{-1} \| \langle x_n, x_n \rangle, \quad \forall n \in \mathbb{N},$$

so  $\langle x_n, x_n \rangle$  is invertible for all  $n \in \mathbb{N}$ .

Finally, if  $\sum_{n=1}^{\infty} x_n a_n = 0$  for some sequence  $(a_n)_n$  in  $A$ , then we also have  $(U^* U)^{-\frac{1}{2}} (\sum_{n=1}^{\infty} x_n a_n) = 0$ , i.e.,  $\sum_{n=1}^{\infty} y_n a_n = 0$ . Then for all  $m \in \mathbb{N}$  we have

$$0 = \langle y_m, \sum_{n=1}^{\infty} y_n a_n \rangle = \sum_{n=1}^{\infty} \langle y_m, y_n a_n \rangle = \sum_{n=1}^{\infty} \delta_{mn} a_n = a_m,$$

and (c) is proved.  $\square$

We conclude this section with a remark concerning finite frames and outer frames and their duals.

*Remark 4.21.* Let  $(x_n)_{n=1}^N$  be a frame or an outer frame for a Hilbert  $C^*$ -module  $X$ . A frame or an outer frame  $(y_n)_{n=1}^N$  is said to be dual to  $(x_n)_{n=1}^N$  if  $\sum_{n=1}^N y_n \langle x_n, x \rangle = x$  for all  $x \in X$ .

Observe that the analysis operator  $U$  of  $(x_n)_{n=1}^N$  takes values in  $A^N$ . It is easy to see that, with this difference, i.e., with  $A^N$  playing the role of  $\ell^2(A)$ , all the preceding results from this section survive. In particular, one can show that, for  $N \in \mathbb{N}$ , Hilbert  $C^*$ -modules  $A^N$  have properties analogous to those of  $\ell^2(A)$  discussed in Remark 4.19. We omit the details.

## 5. PERTURBATIONS AND TIGHT APPROXIMATIONS OF FRAMES

In this section we study neighborhoods of frames and outer frames. In fact, our results will be stated in terms of neighborhoods of the corresponding analysis operators.

There are several important results concerning perturbations of frames for Hilbert spaces (see [7] and references therein). Perturbations of frames for Hilbert  $C^*$ -modules are considered in [14]. A remarkable property of any frame  $(x_n)_n$  for a Hilbert space is that one can always find a neighborhood of  $(x_n)_n$  (defined in terms of  $\ell^2$ -distance of sequences or in terms of the distance of analysis/synthesis operators) such that each sequence belonging to that neighborhood (i.e., sufficiently close to  $(x_n)_n$ ) is also a frame. As usual, the situation is more complicated in the modular context.

We begin with an example which shows that in any Hilbert  $C^*$ -module  $X$  such that  $M(X) \neq X$  we can find a frame for  $X$  such that any neighborhood of its analysis operator contains an operator that is not the analysis operator of any frame for  $X$ .

*Example 5.1.* Let  $X$  be a Hilbert  $A$ -module such that  $M(X) \neq X$  and  $v \in M(X) \setminus X$  such that  $\|v\| = 1$ . Take arbitrary  $\varepsilon > 0$ .

Let  $(x_n)_n$  be a frame for  $X$ . Then the sequence  $0, x_1, x_2, x_3, \dots$  is also a frame for  $X$ , and its analysis operator  $U \in \mathbb{B}(X, \ell^2(A))$  is given by  $Ux = (0, \langle x_1, x \rangle, \langle x_2, x \rangle, \dots)$ . Further, the sequence  $\varepsilon v, x_1, x_2, x_3, \dots$  is an outer frame for  $X$  and its analysis operator  $V \in \mathbb{B}(X, \ell^2(A))$  is given by  $Vx = (\varepsilon \langle v, x \rangle, \langle x_1, x \rangle, \langle x_2, x \rangle, \dots)$ . Observe that the operator  $V$ , being the analysis operator of an outer frame for  $X$ , is not the analysis operator of any frame for  $X$ . On the other hand,

$$\|U - V\| = \sup\{\varepsilon \|\langle v, x \rangle\| : x \in X, \|x\| \leq 1\} \leq \varepsilon \|v\| = \varepsilon.$$

The above example suggests that, as in the preceding section, in order to obtain analogues of the classical results, one should include outer frames into the consideration.

We restrict our discussion to *infinite sequences*. Thereby, we shall understand that finite frames  $(x_n)_{n=1}^N$  are extended to infinite sequences by adding infinitely many zero vectors.

**Theorem 5.2.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$  and the optimal lower frame bound  $A$ . Suppose that  $V \in \mathbb{B}(X, \ell^2(A))$  satisfies  $\|U - V\| < \sqrt{A}$ . Then  $V$  is the analysis operator of a frame or an outer frame  $(y_n)_n$  for  $X$  such that*

$$\|x_n - y_n\| \leq \|U - V\| < \sqrt{A}, \quad \forall n \in \mathbb{N}.$$

*Proof.* Let  $\|U - V\| = m < \sqrt{A}$ . Then

$$\|Vx\| \geq \|Ux\| - \|Ux - Vx\| \geq \sqrt{A}\|x\| - m\|x\| = (\sqrt{A} - m)\|x\|$$

for all  $x \in X$ . Thus,  $V$  is bounded from below and consequently,  $V^* \in \mathbb{B}(\ell^2(A), X)$  is a surjection. By Theorem 3.19,  $V^*$  is the synthesis operator of a frame or an outer frame  $(y_n)_n$  for  $X$  defined by  $y_n = (V_M)^* e^{(n)}$ ,  $n \in \mathbb{N}$ . Then, using Remark 3.2(d), for each  $n \in \mathbb{N}$  we have

$$\begin{aligned} \|x_n - y_n\| &= \|(U_M - V_M)^* e^{(n)}\| = \|(U^* - V^*)_M e^{(n)}\| \\ &\leq \|(U^* - V^*)_M\| = \|U^* - V^*\| \\ &= \|U - V\| < \sqrt{A}. \end{aligned}$$

Observe that the extended operators  $U_M$  and  $V_M$  coincide with  $U$  and  $V$ , respectively, when  $A$  is unital. On the other hand, if  $A$  is non-unital and  $x_n$  or  $y_n$  belongs to  $M(X) \setminus X$  for some  $n$ , then the expression  $\|x_n - y_n\|$  is computed in the multiplier module  $M(X)$ .  $\square$

Let us first note an easy consequence of this result. A similar result appeared in Theorem 3.16. of [15].

**Corollary 5.3.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$  and the optimal lower frame bound  $A$ . If  $\|x_j\| < \sqrt{A}$  for some  $j$ , then  $(x_n)_{n \neq j}$  is a frame or an outer frame for  $X$ .*

*Proof.* Let us define a sequence  $(y_n)_n$  as  $y_j = 0$  and  $y_n = x_n$  for  $n \neq j$ . Since  $(x_n)_n$  is a frame or an outer frame for  $X$ ,  $(y_n)_n$  is a Bessel sequence or an outer Bessel sequence. Let  $V \in \mathbb{B}(X, \ell^2(A))$  be the analysis operator associated to  $(y_n)_n$ . Since  $\|(U - V)x\| = \|\langle x_j, x \rangle\|$  for all  $x \in X$ , we have  $\|U - V\| = \|x_j\| < \sqrt{A}$ , so by Theorem 5.2,  $(y_n)_n$  is a frame or an outer frame for  $X$ . Then obviously,  $(x_n)_{n \neq j}$  is also a frame or an outer frame for  $X$ .  $\square$

*Remark 5.4.* The open ball from Theorem 5.2 is the largest open ball around  $U$  with that property. Indeed, let us consider an orthonormal basis  $(\epsilon_n)_n$  for a Hilbert space  $H$  as a frame for  $H$ ; the analysis operator  $U$  is then an isometry and the optimal lower bound is  $A = 1$ . If we denote by  $V$  the

analysis operator of the Bessel sequence  $\{0\} \cup (\epsilon_n)_{n \geq 2}$  (which is not a frame for  $H$ ), then  $\|V - U\| = 1 = \sqrt{A}$ , so the boundary of the open ball around  $U$  with the radius  $\sqrt{A}$  contains an operator which is not the analysis operator of any frame (or outer frame) for  $H$ .

At this point we need to make a comment on Theorem 3.2 from [14]. The second statement of that theorem may be rephrased as follows:

*Let  $(x_n)_n$  be a frame for a Hilbert  $C^*$ -module  $X$  over a unital  $C^*$ -algebra  $A$  with the analysis operator  $U$  and the frame bounds  $A$  and  $B$ . Suppose that  $(y_n)_n$  is a sequence in  $X$  for which there exist constants  $\lambda_1, \lambda_2, \mu \geq 0$  with the properties*

$$(40) \quad \max\{\lambda_1 + \frac{\mu}{\sqrt{A}}, \lambda_2\} < 1,$$

and

$$(41) \quad \left\| \sum_{n=1}^N (x_n - y_n) a_n \right\| \leq \lambda_1 \left\| \sum_{n=1}^N x_n a_n \right\| + \lambda_2 \left\| \sum_{n=1}^N y_n a_n \right\| + \mu \left\| \sum_{n=1}^N a_n^* a_n \right\|^{\frac{1}{2}},$$

for all finite sequences  $(a_1, \dots, a_N, 0, 0, \dots) \in c_{00}(A)$ . Then  $(y_n)_n$  is also a frame for  $X$ .

Clearly, one could easily deduce our Proposition 5.2 from this statement, at least in the unital case. Indeed, suppose we are given an operator  $V \in \mathbb{B}(X, \ell^2(A))$  such that  $\|U - V\| = \mu < \sqrt{A}$ . Put  $y_n = V^* e^{(n)}$ ,  $n \in \mathbb{N}$ . Then, obviously, we have for each  $(a_1, \dots, a_N, 0, 0, \dots) \in c_{00}(A)$ ,

$$\left\| \sum_{n=1}^N (x_n - y_n) a_n \right\| = \|(U^* - V^*)(a_1, \dots, a_N, 0, 0, \dots)\| \leq \mu \left\| \sum_{n=1}^N a_n^* a_n \right\|^{\frac{1}{2}},$$

which means that the sequence  $(y_n)_n$  satisfies (41) with  $\lambda_1 = \lambda_2 = 0$ . Since  $\mu < \sqrt{A}$ , we also have (40); thus, by applying the above statement one could conclude that  $(y_n)_n$  is a frame for  $X$ .

However, there is a gap in the proof of the above statement (i.e., the second part of Theorem 3.2. from [14]) and it is not clear how one can fix the proof presented there. Namely, that proof uses Lemma 2.7 and Proposition 2.8. from [14] which, as we have seen in our Example 1.5 and Remark 1.6, fail to be generally true. It seems that in order to obtain a result as in aforementioned Theorem 3.2 from [14], one should additionally include in the hypothesis that the sequence  $(y_n)_n$  is Bessel.

We proceed with a remark that is known, but which we include for convenience of the reader.

*Remark 5.5.* Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with optimal frame bounds  $A$  and  $B$ . Let us describe  $A$  and  $B$  in terms of the associated analysis operator  $U$ .

First, by Theorem 2.8 and Remark 2.9 from [19] we conclude that the optimal upper frame bound  $B$  satisfies

$$(42) \quad \sqrt{B} = \|U\| = \min\{M \geq 0 : \|Ux\| \leq M\|x\|, x \in X\}.$$

Further, writing the relation  $\langle Ux, Ux \rangle \geq A\langle x, x \rangle, x \in X$ , in an equivalent form  $\langle (U^*U)^{-\frac{1}{2}}x, (U^*U)^{-\frac{1}{2}}x \rangle \leq \frac{1}{A}\langle x, x \rangle, x \in X$  (obtained by replacing  $x$  with  $(U^*U)^{-\frac{1}{2}}x$ ), and then applying (42) we get

$$\begin{aligned} \frac{1}{\sqrt{A}} &= \|(U^*U)^{-\frac{1}{2}}\| \\ &= \min\{M \geq 0 : \|(U^*U)^{-\frac{1}{2}}x\| \leq M\|x\|, x \in X\} \\ &\quad (\text{replace } x \text{ with } (U^*U)^{\frac{1}{2}}x \text{ and apply } \|Ux\| = \|(U^*U)^{\frac{1}{2}}x\|) \\ &= \min\{M \geq 0 : \|Ux\| \geq \frac{1}{M}\|x\|, x \in X\} \\ &= (\max\{m \geq 0 : \|Ux\| \geq m\|x\|, x \in X\})^{-1}. \end{aligned}$$

Therefore,

$$(43) \quad \sqrt{A} = \|(U^*U)^{-\frac{1}{2}}\|^{-1} = \max\{m \geq 0 : \|Ux\| \geq m\|x\|\}.$$

The following corollary provides another useful property of the open ball with the center in  $U$  that is considered in Theorem 5.2.

**Corollary 5.6.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$  and the optimal lower frame bound  $A$ . Suppose that  $V \in \mathbb{B}(X, \ell^2(A))$  satisfies  $\|U - V\| < \sqrt{A}$ . Then  $V^*U \in \mathbb{B}(X)$  is an invertible operator.*

*Proof.* Put again  $\|U - V\| = m < \sqrt{A}$ . Then

$$\begin{aligned} \|x - V^*U(U^*U)^{-1}x\| &= \|U^*U(U^*U)^{-1}x - V^*U(U^*U)^{-1}x\| \\ &= \|(U^* - V^*)(U(U^*U)^{-1}x)\| \\ &\leq m\|U(U^*U)^{-1}x\| \end{aligned}$$

for all  $x \in X$ . By taking the supremum over the unit ball in  $X$  we get

$$\|I - V^*U(U^*U)^{-1}\| \leq m\|U(U^*U)^{-1}\|.$$

Since  $\|U(U^*U)^{-1}\|^2 = \|(U(U^*U)^{-1})^*(U(U^*U)^{-1})\| = \|(U^*U)^{-1}\| = \frac{1}{A}$ , we have

$$\|I - V^*U(U^*U)^{-1}\| \leq \frac{m}{\sqrt{A}} < 1.$$

This shows that  $V^*U(U^*U)^{-1}$  is an invertible operator. In particular,  $V^*U$  is invertible as well.  $\square$

*Remark 5.7.* We say that frames or outer frames  $(x_n)_n$  and  $(y_n)_n$  for a Hilbert  $C^*$ -module  $X$  with the analysis operators  $U$  and  $V$  are *pseudodual* if  $V^*U$  is an invertible operator. When this is the case, we have, for each  $x \in X$ ,

$$x = U^*V((U^*V)^{-1}x) = \sum_{n=1}^{\infty} x_n \langle y_n, (U^*V)^{-1}x \rangle = \sum_{n=1}^{\infty} x_n \langle (V_M^*U_M)^{-1}y_n, x \rangle.$$

This shows that  $(x_n)_n$  and  $((V_M^*U_M)^{-1}y_n)_n$  are dual to each other. In an analogous way we conclude that  $(y_n)_n$  and  $((U_M^*V_M)^{-1}x_n)_n$  are also dual to each other.

Given a frame or an outer frame  $(x_n)_n$  for a Hilbert  $A$ -module  $X$ , we now want to find a Parseval frame for  $X$  closest to  $(x_n)_n$ , again measured in terms of distance of the corresponding analysis operators. As one might expect, a solution is the Parseval frame canonically associated with  $(x_n)_n$ , i.e.,  $(y_n)_n$ , where  $y_n = (U_M^*U_M)^{-\frac{1}{2}}x_n$ ,  $n \in \mathbb{N}$ , and  $U$  is the analysis operator of  $(x_n)_n$ . Recall that  $(y_n)_n$  is outer if and only if  $(x_n)_n$  is outer; nevertheless, its analysis operator is always equal to  $U(U^*U)^{-\frac{1}{2}}$ .

**Proposition 5.8.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$  and the optimal frame bounds  $A$  and  $B$ . If  $(y_n)_n$  is the Parseval frame canonically associated with  $(x_n)_n$ , then its analysis operator  $U(U^*U)^{-\frac{1}{2}}$  satisfies*

$$\left\| U - U(U^*U)^{-\frac{1}{2}} \right\| = \max \left\{ 1 - \sqrt{A}, \sqrt{B} - 1 \right\}.$$

*If  $(y_n)_n$  is any Parseval frame or outer Parseval frame for  $X$ , then its analysis operator  $V$  satisfies*

$$\|U - V\| \geq \max \left\{ 1 - \sqrt{A}, \sqrt{B} - 1 \right\}.$$

*Proof.* First, we have

$$\begin{aligned} \left\| U - U(U^*U)^{-\frac{1}{2}} \right\| &= \left\| \left( U - U(U^*U)^{-\frac{1}{2}} \right)^* \left( U - U(U^*U)^{-\frac{1}{2}} \right) \right\|^{\frac{1}{2}} \\ &= \left\| \left( (U^*U)^{-\frac{1}{2}} - I \right) U^*U \left( (U^*U)^{-\frac{1}{2}} - I \right) \right\|^{\frac{1}{2}} \\ &= \left\| \left( I - (U^*U)^{\frac{1}{2}} \right)^2 \right\|^{\frac{1}{2}} \\ &= \left\| I - (U^*U)^{\frac{1}{2}} \right\| \\ &= \max \left\{ |1 - \sqrt{A}|, |1 - \sqrt{B}| \right\} \\ &= \max \left\{ 1 - \sqrt{A}, \sqrt{B} - 1 \right\}. \end{aligned}$$

To prove the second assertion, suppose that  $(y_n)_n$  is a Parseval frame or an outer Parseval frame for  $X$ . Then its analysis operator  $V \in \mathbb{B}(X, \ell^2(\mathbf{A}))$  is an isometry, so we have

$$\|Ux\| \geq \|Vx\| - \|Vx - Ux\| = \|x\| - \|Vx - Ux\| \geq (1 - \|U - V\|)\|x\|$$

for all  $x \in X$ . By (43) we get  $\sqrt{A} \geq 1 - \|U - V\|$ , that is,  $\|U - V\| \geq 1 - \sqrt{A}$ .

On the other hand,  $\sqrt{B} = \|U\| \leq \|U - V\| + \|V\| = \|U - V\| + 1$ , wherefrom  $\|U - V\| \geq \sqrt{B} - 1$ . Therefore,  $\|U - V\| \geq \max\{1 - \sqrt{A}, \sqrt{B} - 1\}$ .  $\square$

In a similar fashion we can find the distance of a given frame or outer frame  $(x_n)_n$  for  $X$  with the optimal bounds  $A$  and  $B$  to the set of all tight frames and outer tight frames for  $X$ . It turns out that this distance is equal to  $\frac{\sqrt{B} - \sqrt{A}}{2}$ . For Hilbert space frames this question was discussed in [11, Proposition 5.4].

**Proposition 5.9.** *Let  $(x_n)_n$  be a frame or an outer frame for a Hilbert  $A$ -module  $X$  with the analysis operator  $U$  and the optimal frame bounds  $A$  and  $B$ . Let  $V_0 = \frac{\sqrt{A} + \sqrt{B}}{2} U(U^*U)^{-\frac{1}{2}}$ . Then  $V_0$  is the analysis operator of a  $\left(\frac{\sqrt{A} + \sqrt{B}}{2}\right)^2$ -tight frame or outer frame for  $X$  for which*

$$\|U - V_0\| = \frac{\sqrt{B} - \sqrt{A}}{2}.$$

If  $(y_n)_n$  is any tight frame or outer frame for  $X$  with the analysis operator  $V$ , then

$$\|U - V\| \geq \frac{\sqrt{B} - \sqrt{A}}{2}.$$

*Proof.* Suppose first that  $(y_n)_n$  is a frame or an outer frame for  $X$  whose analysis operator  $V$  is of the form  $V^*V = \lambda^2 I$  for some scalar  $\lambda > 0$ . Then, as in the preceding proof, we have

$$\|Ux\| \geq \|Vx\| - \|Vx - Ux\| = \lambda\|x\| - \|Vx - Ux\| \geq (\lambda - \|U - V\|)\|x\|$$

for all  $x \in X$ , so by (43) we get  $\sqrt{A} \geq \lambda - \|U - V\|$ , that is,

$$(44) \quad \|U - V\| \geq \lambda - \sqrt{A}.$$

On the other side,  $\sqrt{B} = \|U\| \leq \|U - V\| + \|V\| = \|U - V\| + \lambda$ ; thus,

$$(45) \quad \|U - V\| \geq \sqrt{B} - \lambda.$$

Adding (44) and (45) we get  $\|U - V\| \geq \frac{\sqrt{B} - \sqrt{A}}{2}$ .

Consider now  $V_0 = \frac{\sqrt{A} + \sqrt{B}}{2} U(U^*U)^{-\frac{1}{2}}$ . An easy verification shows that  $V_0$  is the analysis operator of a frame or an outer frame  $(y_n)_n$  given by  $y_n = \frac{\sqrt{A} + \sqrt{B}}{2} (U_M^* U_M)^{-\frac{1}{2}} x_n$  for  $n \in \mathbb{N}$ . Since  $V_0^* V_0 = \left(\frac{\sqrt{A} + \sqrt{B}}{2}\right)^2 I$ , this is



a tight frame or outer frame. Then, repeating (each particular step of) the computation from the beginning of the preceding proof we get

$$\begin{aligned} \|U - V_0\| &= \left\| U - \frac{\sqrt{A} + \sqrt{B}}{2} U(U^*U)^{-\frac{1}{2}} \right\| \\ &= \max \left\{ \frac{\sqrt{A} + \sqrt{B}}{2} - \sqrt{A}, \sqrt{B} - \frac{\sqrt{A} + \sqrt{B}}{2} \right\} \\ &= \frac{\sqrt{B} - \sqrt{A}}{2}. \end{aligned}$$

□

*Remark 5.10.* Observe that "the best tight approximation", namely the frame or an outer frame from the preceding proposition, is actually Parseval if and only if  $\sqrt{A} + \sqrt{B} = 2$ . If this is the case the resulting distance is equal to  $1 - \sqrt{A} = \sqrt{B} - 1$  and this result is, for  $\sqrt{A} + \sqrt{B} = 2$ , in accordance with Proposition 5.8.

## 6. FINITE EXTENSIONS OF BESSEL SEQUENCES

Finite extensions of Bessel sequences to frames in Hilbert spaces are recently discussed in [3]. In this section we discuss the same problem in modular context.

First note that each Bessel sequence in an AFG Hilbert  $C^*$ -module  $X$  admits a finite extension to a frame: given a Bessel sequence (finite or infinite) in  $X$  it suffices to extend it by any finite set of generators for  $X$ . Thus, here we are interested primarily in countably generated Hilbert  $C^*$ -modules which are not AFG.

As before, our discussion will include both frames and outer frames. We shall first characterize (again in terms of analysis operators) those Bessel sequences and outer Bessel sequences in a Hilbert  $C^*$ -module  $X$  that admit finite extensions to frames or outer frames for  $X$ . After that, more specifically, we shall describe Bessel sequences and outer Bessel sequences which allow finite extensions to Parseval frames or outer Parseval frames.

A related, but more restrictive question we address is the following: given a Bessel sequence in  $X$ , does there exist its finite extension to a frame for  $X$ ? We shall find necessary and sufficient conditions under which one can extend a given Bessel sequence to a frame by adding finitely many elements of  $X$ . This is, indeed, a stronger property; we shall see in Example 6.3, that there are Bessel sequences that do not admit finite extensions to frames, but which do admit finite extensions (by elements of  $M(X) \setminus X$ ) to outer frames.

We begin with our most general result on finite extensions.

**Theorem 6.1.** *Let  $(x_n)_{n=1}^\infty$  be a Bessel or an outer Bessel sequence in a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . Then there is a finite extension of  $(x_n)_n$  to a frame or an outer frame for  $X$  if and only if there exist  $V \in \mathbb{B}(X, \ell^2(A))$  and  $\theta \in \mathbb{F}(M(X))$  such that  $I - V^*U = \theta|_X$ .*

*Proof.* Suppose that a Bessel sequence  $(x_n)_n$  admits a finite extension to a frame or an outer frame for  $X$ . Let  $f_1, \dots, f_N \in M(X)$ ,  $N \in \mathbb{N}$ , be such that  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$  is a frame or an outer frame for  $X$ . Let  $U_1, F \in \mathbb{B}(X, \ell^2(A))$  be analysis operators of Bessel or outer Bessel sequences  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$  and  $(f_n)_{n=1}^N \cup (0)_{n=1}^\infty$  respectively (so, in the later case we have  $f_n$ 's followed by infinitely many zeros). Obviously,  $U_1 = F + S^N U$ , where  $S$  denotes the unilateral shift on  $\ell^2(A)$ .

Let us take any frame or outer frame dual to  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$ , and write it, for convenience, as  $(g_n)_{n=1}^N \cup (y_n)_{n=1}^\infty$ . Let  $G, V, V_1 \in \mathbb{B}(X, \ell^2(A))$  be the analysis operators of the Bessel sequences or outer Bessel sequences  $(g_n)_{n=1}^N \cup (0)_{n=1}^\infty$ ,  $(y_n)_{n=1}^\infty$ , and  $(g_n)_{n=1}^N \cup (y_n)_{n=1}^\infty$ , respectively. Again,  $V_1 = G + S^N V$ .

Since  $(g_n)_{n=1}^N \cup (y_n)_{n=1}^\infty$  and  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$  are dual to each other, it follows  $V_1^* U_1 = I$ . Since, obviously,  $G^* S^N = 0$  and  $(S^N)^* F = 0$ , we have

$$\begin{aligned} I &= (G + S^N V)^*(F + S^N U) \\ &= G^* F + V^*(S^N)^* F + G^* S^N U + V^*(S^N)^* S^N U \\ &= G^* F + V^* U, \end{aligned}$$

that is,  $I - V^* U = G^* F$ . Let  $\theta \in \mathbb{F}(M(X))$  be defined as  $\theta = \sum_{n=1}^N \theta_{g_n, f_n}$ . Then

$$G^* F(x) = \sum_{n=1}^N g_n \langle f_n, x \rangle = \sum_{n=1}^N \theta_{g_n, f_n}(x) = \theta(x), \quad \forall x \in X,$$

so we conclude that  $I - V^* U = \theta|_X$ .

Conversely, suppose there is  $V \in \mathbb{B}(X, \ell^2(A))$  and  $\theta \in \mathbb{F}(M(X))$  such that  $I - V^* U = \theta|_X$ . Let  $f_1, \dots, f_N, g_1, \dots, g_N \in M(X)$  be such that  $\theta = \sum_{n=1}^N \theta_{g_n, f_n}$ . By Corollary 3.21 there is a Bessel sequence or an outer Bessel sequence  $(y_n)_n$  such that  $V$  is its analysis operator. Then the sequences  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$  and  $(g_n)_{n=1}^N \cup (y_n)_{n=1}^\infty$  are also Bessel or outer Bessel sequences. Let  $F, G, U_1, V_1$  be as before. The same computation shows that

$$V_1^* U_1 = G^* F + V^* U = \theta|_X + V^* U = I,$$

so, by Lemma 4.3,  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$  and  $(g_n)_{n=1}^N \cup (y_n)_{n=1}^\infty$  are frames or outer frames for  $X$ .  $\square$

In the same way one proves the following corollary which concerns finite extensions of a Bessel sequence by elements of the original Hilbert  $C^*$ -module  $X$  (i.e., without using elements from  $M(X) \setminus X$ ) in which case we end up with a frame for  $X$ .

**Corollary 6.2.** *Let  $(x_n)_{n=1}^\infty$  be a Bessel sequence in a Hilbert  $A$ -module  $X$  with the analysis operator  $U$ . Then there is a finite extension of  $(x_n)_n$  to a frame for  $X$  if and only if there exists  $V \in \mathbb{B}(X, \ell^2(A))$  such that  $I - V^*U \in \mathbb{F}(X)$ .*

The above property of a Bessel sequence is more restrictive than that from Theorem 6.1. To see this, we demonstrate an example of a Bessel sequence that does not allow a finite extension to a frame, but which does have (many) finite extensions to outer frames.

*Example 6.3.* Take a separable infinite-dimensional Hilbert space  $H$  and consider  $X = \mathbb{K}(H)$  as a Hilbert  $\mathbb{K}(H)$ -module in the standard way. Let  $(\epsilon_n)_n$  be an orthonormal basis for  $H$ . For each  $n \in \mathbb{N}$  denote by  $e_n$  the orthogonal projection to  $\text{span}\{\epsilon_n\}$ . Further, put  $H_1 = \overline{\text{span}}\{\epsilon_{2n-1} : n \in \mathbb{N}\}$  and  $H_2 = \overline{\text{span}}\{\epsilon_{2n} : n \in \mathbb{N}\}$ . If we denote by  $p_1$  and  $p_2$  the corresponding orthogonal projections then, obviously,  $p_1 + p_2 = e$ , where  $e$  denotes the identity operator on  $H$ .

Consider now the sequence  $(x_n)_n$  in  $X$  defined by  $x_n = e_{2n}$  for all  $n \in \mathbb{N}$ . For each  $a \in X$  we have

$$\sum_{n=1}^{\infty} \langle a, x_n \rangle \langle x_n, a \rangle = \sum_{n=1}^{\infty} a^* e_{2n} a = a^* p_2 a,$$

with the convergence in norm, so by Proposition 2.1,  $(x_n)_n$  is a Bessel sequence in  $X$ . Let  $U$  be its analysis operator.

Let us first show that  $(x_n)_n$  does not admit a finite extension to a frame for  $X$ . To prove this, suppose the opposite. Then by Corollary 6.2, there exist  $f_1, \dots, f_N, g_1, \dots, g_N \in X$  for some  $N \in \mathbb{N}$ , and an operator  $V \in \mathbb{B}(X, \ell^2(\mathbb{K}(H)))$  (which is the analysis operator of a Bessel or an outer Bessel sequence  $(y_n)_n$  in  $X$ ) such that  $I - V^*U = \sum_{n=1}^N \theta_{f_n, g_n}$ . This means that

$$a - \sum_{n=1}^{\infty} y_n \langle x_n, a \rangle = \sum_{n=1}^N f_n \langle g_n, a \rangle, \quad \forall a \in X.$$

Denote  $b = \sum_{n=1}^N f_n g_n^*$  and observe that  $b \in X = \mathbb{K}(H)$ . Now the preceding equality can be rewritten as

$$a - \sum_{n=1}^{\infty} y_n e_{2n} a = ba, \quad \forall a \in X.$$

In particular, if  $a$  is any operator in  $\mathbb{K}(H)$  whose range is contained in  $H_1$ , we have  $a = ba$ . This in turn implies  $b\epsilon_{2n-1} = \epsilon_{2n-1}$  for all  $n \in \mathbb{N}$  wherefrom we conclude that a closed infinite dimensional subspace  $H_1$  of  $H$  is contained in the range of a compact operator  $b$ , which is a contradiction.

Next we show that  $(x_n)_n$  can be extended to an outer frame for  $X$  by adding a single vector from  $M(X) \setminus X$ . Namely, if  $c \in M(X) = \mathbb{B}(H)$  is

invertible, then  $cc^* \geq \frac{1}{\|c^{-1}\|^2}e$ , so

$$\langle a, c \rangle \langle c, a \rangle + \sum_{n=1}^{\infty} \langle a, x_n \rangle \langle x_n, a \rangle \geq \langle a, c \rangle \langle c, a \rangle = a^* cc^* a \geq \frac{1}{\|c^{-1}\|^2} a^* a$$

for all  $a \in X$ . Thus,  $c, x_1, x_2, \dots$  is an outer frame for  $X$ .

Moreover,  $(x_n)_n$  can be extended to an outer Parseval frame for  $X$ , again by adding just one vector from  $M(X) \setminus X$ . Indeed, we have for each  $x \in X$

$$\langle a, p_1 \rangle \langle p_1, a \rangle + \sum_{n=1}^{\infty} a^* e_{2n} a = a^* p_1 a + a^* p_2 a = a^* a$$

so the sequence  $p_1, x_1, x_2, x_3, \dots$  is an outer Parseval frame for  $X$ .

Our next goal is to describe those Bessel sequences that admit finite extensions to Parseval frames. Note that this question, in contrast to the preceding one, is non-trivial even for AFG Hilbert  $C^*$ -modules. First we need some auxiliary results. We begin with a lemma which is a variant of Lemma 5.5.4 from [22].

**Lemma 6.4.** *Let  $X$  be a Hilbert  $A$ -module. Let  $x \in X$  and  $a \in A$  be such that  $0 \leq a \leq \langle x, x \rangle$ . Then there exists  $z \in X$  such that  $a = \langle z, z \rangle$ .*

*Proof.* Let  $v \in X$  be such that  $x = v \langle v, v \rangle$ . Let  $y = v \langle v, v \rangle^{\frac{1}{4}}$ . Then

$$\langle y, y \rangle^2 = \langle v, v \rangle^{\frac{1}{4}} \langle v, v \rangle \langle v, v \rangle^{\frac{1}{4}} \langle v, v \rangle^{\frac{1}{4}} \langle v, v \rangle \langle v, v \rangle^{\frac{1}{4}} = \langle v, v \rangle^3 = \langle x, x \rangle.$$

Write  $\langle y, y \rangle = c$ . Then we have  $0 \leq a \leq c^2$ . Put

$$b_n = \left(c + \frac{1}{n}e\right)^{-\frac{1}{2}} a^{\frac{1}{2}}, \quad n \in \mathbb{N}.$$

Here  $e$  denotes the unit in  $A$  or in  $\tilde{A}$ , but in both cases  $b_n \in A$  for all  $n$ . Then for all  $m, n \in \mathbb{N}$ ,  $n \geq m$  we have

$$\begin{aligned} \|b_n - b_m\|^2 &= \|(b_n - b_m)(b_n - b_m)^*\| \\ &= \left\| \left( \left(c + \frac{1}{n}e\right)^{-\frac{1}{2}} - \left(c + \frac{1}{m}e\right)^{-\frac{1}{2}} \right) a \left( \left(c + \frac{1}{n}e\right)^{-\frac{1}{2}} - \left(c + \frac{1}{m}e\right)^{-\frac{1}{2}} \right) \right\| \\ &\leq \left\| \left( \left(c + \frac{1}{n}e\right)^{-\frac{1}{2}} - \left(c + \frac{1}{m}e\right)^{-\frac{1}{2}} \right) c^2 \left( \left(c + \frac{1}{n}e\right)^{-\frac{1}{2}} - \left(c + \frac{1}{m}e\right)^{-\frac{1}{2}} \right) \right\| \\ &= \left\| c \left( \left(c + \frac{1}{n}e\right)^{-\frac{1}{2}} - \left(c + \frac{1}{m}e\right)^{-\frac{1}{2}} \right) \right\|^2. \end{aligned}$$

The sequence  $(f_n)_n$ ,  $f_n(t) = t(t + \frac{1}{n})^{-\frac{1}{2}}$  is an increasing sequence of positive continuous functions that converges pointwise for  $t \in [0, \|c\|]$  to the

continuous function  $f(t) = \sqrt{t}$ . By Dini's theorem  $(f_n)_n$  converges to  $f$  uniformly on  $[0, \|c\|]$ ; hence,

$$(46) \quad \lim_{n \rightarrow \infty} c \left( c + \frac{1}{n} e \right)^{-\frac{1}{2}} = c^{\frac{1}{2}}.$$

Now the above computation shows that  $(b_n)_n$  is a Cauchy sequence in  $\mathbf{A}$ . Put  $b = \lim_{n \rightarrow \infty} b_n$ . Then we have  $\langle y, y \rangle^{\frac{1}{2}} b = \lim_{n \rightarrow \infty} \langle y, y \rangle^{\frac{1}{2}} b_n$  and  $b^* \langle y, y \rangle^{\frac{1}{2}} = \lim_{n \rightarrow \infty} b_n^* \langle y, y \rangle^{\frac{1}{2}}$  which implies  $b^* \langle y, y \rangle b = \lim_{n \rightarrow \infty} b_n^* \langle y, y \rangle b_n$ , that is,

$$(47) \quad \langle yb, yb \rangle = \lim_{n \rightarrow \infty} \langle yb_n, yb_n \rangle.$$

On the other hand,

$$\begin{aligned} \|a - \langle yb_n, yb_n \rangle\| &= \|a - b_n^* \langle y, y \rangle b_n\| \\ &= \left\| a - a^{\frac{1}{2}} \left( c + \frac{1}{n} e \right)^{-\frac{1}{2}} c \left( c + \frac{1}{n} e \right)^{-\frac{1}{2}} a^{\frac{1}{2}} \right\| \\ &= \left\| \left( a^{\frac{1}{2}} \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right)^{\frac{1}{2}} \right) \left( a^{\frac{1}{2}} \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right)^{\frac{1}{2}} \right)^* \right\| \\ &= \left\| \left( a^{\frac{1}{2}} \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right)^{\frac{1}{2}} \right)^* \left( a^{\frac{1}{2}} \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right)^{\frac{1}{2}} \right) \right\| \\ &= \left\| \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right)^{\frac{1}{2}} a \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right)^{\frac{1}{2}} \right\| \\ &\leq \left\| c^2 \left( e - c \left( c + \frac{1}{n} e \right)^{-1} \right) \right\| \\ &= \left\| c^2 - c^3 \left( c + \frac{1}{n} e \right)^{-1} \right\|. \end{aligned}$$

It follows from (46) that  $\lim_{n \rightarrow \infty} c^3(c + \frac{1}{n}e)^{-1} = c^2$ . Hence, the above computation shows that  $\lim_{n \rightarrow \infty} \langle yb_n, yb_n \rangle = a$ . This, together with (47), gives us  $\langle yb, yb \rangle = a$ . Put  $z = yb$ .  $\square$

**Proposition 6.5.** *Let  $X$  be a Hilbert  $\mathbf{A}$ -module and  $a \in \mathbf{A}, a \geq 0$ , such that  $a = \sum_{n=1}^N \langle u_n, v_n \rangle$  for some  $N \in \mathbb{N}$  and  $u_1, \dots, u_N, v_1, \dots, v_N \in X$ . Then there exist  $x_1, \dots, x_N \in X$  such that  $a = \sum_{n=1}^N \langle x_n, x_n \rangle$ .*

*Proof.* It follows from the polarization formula and self-adjointness of  $a$  that

$$4a = \sum_{n=1}^N \langle u_n + v_n, u_n + v_n \rangle - \sum_{n=1}^N \langle u_n - v_n, u_n - v_n \rangle,$$

wherefrom we get

$$(48) \quad a \leq \sum_{n=1}^N \langle \frac{1}{2}u_n + \frac{1}{2}v_n, \frac{1}{2}u_n + \frac{1}{2}v_n \rangle.$$

Let  $X_N = \oplus_{n=1}^N X$  be a direct sum of  $N$  copies of  $X$ , which is a Hilbert  $A$ -module with the inner product defined by  $\langle x, y \rangle = \sum_{n=1}^N \langle x_n, y_n \rangle$ , where  $x = (x_1, \dots, x_N)$  and  $y = (y_1, \dots, y_N)$ .

If we denote  $u = (\frac{1}{2}u_1 + \frac{1}{2}v_1, \dots, \frac{1}{2}u_N + \frac{1}{2}v_N)$ , then (48) reads as  $0 \leq a \leq \langle u, u \rangle$ . By Lemma 6.4, applied in  $X_N$ , there exists  $z \in X_N$  such that  $a = \langle z, z \rangle$ . If we put  $z = (x_1, \dots, x_N)$  then  $a = \sum_{n=1}^N \langle x_n, x_n \rangle$ .  $\square$

Regarding a right Hilbert  $A$ -module  $X$  as a left Hilbert  $\mathbb{K}(X)$ -module we immediately get the following corollary. It refines the statement of Corollary 2.6 in a natural way.

**Corollary 6.6.** *Let  $X$  be a Hilbert  $A$ -module and  $T \in \mathbb{F}(X)$  such that  $T \geq 0$ . Then there exist  $N \in \mathbb{N}$  and  $x_1, \dots, x_N \in X$  such that  $T = \sum_{n=1}^N \theta_{x_n, x_n}$ .*

We are now ready to characterize Bessel sequences and outer Bessel sequences in Hilbert  $C^*$ -modules that admit finite extensions to Parseval frames or outer Parseval frames.

**Theorem 6.7.** *Let  $(x_n)_{n=1}^\infty$  be a Bessel or an outer Bessel sequence in a Hilbert  $A$ -module  $X$  with the analysis operator  $U$  and the optimal Bessel bound  $B$ . Then there is a finite extension of  $(x_n)_n$  to a Parseval or an outer Parseval frame for  $X$  if and only if and there exists  $\theta \in \mathbb{F}(M(X))$  such that  $I - U^*U = \theta|_X$  and  $B \leq 1$ .*

*Proof.* Suppose there exists a finite sequence  $(f_n)_{n=1}^N$  in  $M(X)$  such that  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^\infty$  is a Parseval frame or an outer Parseval frame for  $X$ . Then for every  $x \in X$  it holds

$$(49) \quad \sum_{n=1}^N \langle x, f_n \rangle \langle f_n, x \rangle + \sum_{n=1}^\infty \langle x, x_n \rangle \langle x_n, x \rangle = \langle x, x \rangle.$$

This implies  $\sum_{n=1}^\infty \langle x, x_n \rangle \langle x_n, x \rangle \leq \langle x, x \rangle$  for all  $x \in X$ , so  $B \leq 1$ . Further, if we denote  $\theta = \sum_{n=1}^N \theta_{f_n, f_n} \in \mathbb{F}(M(X))$ , then (49) gives us  $\theta|_X + U^*U = I$ , that is,  $I - U^*U = \theta|_X$ .

Conversely, suppose  $B \leq 1$  and  $I - U^*U = \theta|_X$  for some  $\theta \in \mathbb{F}(M(X))$ . Since  $U^*U \leq B \cdot I = I$  we have  $I - U^*U \geq 0$ . Then its extension  $I_{M(X)} - U_M^*U_M$  is positive and  $I_{M(X)} - U_M^*U_M = \theta$ , so we can apply Corollary 6.6 to  $M(X)$  and  $I_{M(X)} - U_M^*U_M$ . Therefore,  $I_{M(X)} - U_M^*U_M = \sum_{n=1}^N \theta_{f_n, f_n}$  for some  $N \in \mathbb{N}$  and  $f_1, \dots, f_N \in M(X)$ . Now  $I_{M(X)} = U_M^*U_M + \sum_{n=1}^N \theta_{f_n, f_n}$

gives us

$$\sum_{n=1}^N \langle x, f_n \rangle \langle f_n, x \rangle + \sum_{n=1}^{\infty} \langle x, x_n \rangle \langle x_n, x \rangle = \langle x, x \rangle, \quad \forall x \in X.$$

Hence,  $(f_n)_{n=1}^N \cup (x_n)_{n=1}^{\infty}$  is a Parseval frame or an outer Parseval frame for  $X$  depending on whether all  $f_n$ 's are in  $X$  or not.  $\square$

The following corollary is concerned with Bessel sequences and their finite extensions to Parseval frames (so, again, as in Corollary 6.2 we are now interested only in extensions obtained by finitely many elements of the original module  $X$ ). It is convenient to split the statement into two cases: when  $X$  is not AFG, and when  $X$  is AFG.

**Corollary 6.8.** *Let  $X$  be a Hilbert  $A$ -module. Let  $(x_n)_{n=1}^{\infty}$  be a Bessel sequence in  $X$  (either finite or infinite) with the analysis operator  $U$  and the optimal Bessel bound  $B$ .*

- (a) *If  $X$  is not AFG, then  $(x_n)_n$  is finitely extendable to a Parseval frame for  $X$  if and only if  $I - U^*U \in \mathbb{F}(X)$  and  $B = 1$ .*
- (b) *If  $X$  is AFG, then  $(x_n)_n$  is finitely extendable to a Parseval frame for  $X$  if and only if  $B \leq 1$ .*

*Proof.* First, in the same fashion as in the preceding proof we obtain in both cases:  $(x_n)_n$  is finitely extendable to a Parseval frame for  $X$  if and only if  $I - U^*U = \theta \in \mathbb{F}(X)$  and  $B \leq 1$ . We now proceed by specific arguments in each of the above two cases.

(a) Suppose  $X$  is not AFG and  $(x_n)_n$  has a finite extension to a Parseval frame for  $X$ . Then  $I - U^*U$  is non-invertible in  $\mathbb{B}(X)$ , since otherwise we would have

$$I = (I - U^*U)^{-1}(I - U^*U) = (I - U^*U)^{-1}\theta,$$

which, by the ideal property of  $\mathbb{F}(X)$ , gives  $I \in \mathbb{F}(X)$ . But this would imply that  $X$  is AFG, contrary to our assumption. Now, non-invertibility of  $I - U^*U$  means that  $1 \in \sigma(U^*U)$ , so  $B = \|U^*U\| \geq 1$ .

(b) Suppose  $X$  is AFG and  $(x_n)_n$  has a finite extension to a Parseval frame for  $X$ . Here we observe that a general condition  $I - U^*U \in \mathbb{F}(X)$  obtained at the beginning of the proof is automatically satisfied. Indeed, since  $X$  is AFG, we have  $\mathbb{B}(X) = \mathbb{K}(X) = \mathbb{F}(X)$ , so  $I - U^*U \in \mathbb{F}(X)$  for all  $U \in \mathbb{B}(X, \ell^2(A))$ .  $\square$

*Remark 6.9.* Recall that it can happen that  $X$  is not an AFG Hilbert  $A$ -module, but  $M(X)$  is an AFG Hilbert  $M(A)$ -module. (As an example, one can take  $X = A$ , where  $A$  is a non-unital  $C^*$ -algebra). In such cases, each Bessel sequence or an outer Bessel sequence with the optimal Bessel bound  $B < 1$  admits a finite extension to an outer Parseval frame for  $X$ . To see this, denote the corresponding analysis operator by  $U$ . Then, since  $M(X)$  is an AFG module,  $(I - U^*U)_M \in \mathbb{F}(M(X))$ , so the remaining condition

from Theorem 6.7, namely  $I - U^*U = \theta|_X$  for some  $\theta \in \mathbb{F}(M(X))$  is also satisfied.

On the other hand, such sequences cannot allow finite extensions to Parseval frames because of  $B < 1$  (see Corollary 6.8(a)).

We conclude with an application of Corollary 6.8.

**Corollary 6.10.** *Let  $A$  be a non-unital  $\sigma$ -unital  $C^*$ -algebra. Let  $(a_n)_n$  be a sequence in  $A$  such that for all  $a \in A$  the series  $\sum_{n=1}^{\infty} a^* a_n a_n^* a$  converges in norm and  $\|\sum_{n=1}^{\infty} a^* a_n a_n^* a\| \leq \|a\|$ . Then the series  $\sum_{n=1}^{\infty} a_n a_n^*$  strictly converges to an element  $f \in M(A)$  such that  $f \leq e$ . Moreover, if  $e - f \in A$ , then there exists  $b \in A$ ,  $b \geq 0$ , such that the sequence  $(b + \sum_{n=1}^N a_n a_n^*)_N$  is an approximate unit for  $A$ .*

*Proof.* Consider  $A$  as a Hilbert  $A$ -module. Since  $A$  is non-unital and  $\sigma$ -unital,  $A$  is countably generated and not AFG.

First, let us prove that the series  $\sum_{n=1}^{\infty} a_n a_n^*$  is strictly convergent. By the first assumption, the series  $\sum_{n=1}^{\infty} \langle a, a_n \rangle \langle a_n, a \rangle$  converges in norm for every  $a \in A$ . By Theorem 2.1,  $(a_n)_n$  is a Bessel sequence in  $A$ . If  $U \in \mathbb{B}(A, \ell^2(A))$  is its analysis operator, then  $U^*U \in \mathbb{B}(A)$  is given by

$$(50) \quad U^*Ua = \sum_{n=1}^{\infty} a_n \langle a_n, a \rangle = \sum_{n=1}^{\infty} a_n a_n^* a, \quad \forall a \in A.$$

Thus, the series  $\sum_{n=1}^{\infty} a_n a_n^* a$  converges in norm for all  $a \in A$ . By taking adjoints, we conclude that the series  $\sum_{n=1}^{\infty} a^* a_n a_n^*$  is also norm-convergent for all  $a$  in  $A$ . In other words, there exists  $f = (\text{strict}) \sum_{n=1}^{\infty} a_n a_n^* \in M(A)$ . From this we conclude that  $U^*Ua = fa$  for all  $a \in A$ .

Now, assuming that  $e - f \in A$ , we shall prove that  $(a_n)_n$  is a Bessel sequence in  $A$  that admits a finite extension to a Parseval frame for  $A$ .

First, recall that  $\mathbb{K}(A) = \mathbb{F}(A)$ . Therefore, the equality  $(I - U^*U)a = (e - f)a$ ,  $a \in A$ , since  $e - f \in A$ , implies that  $I - U^*U \in \mathbb{F}(A)$ .

Since  $A$  is non-unital, each operator from  $\mathbb{F}(A)$  is non-invertible; in particular,  $I - U^*U$  is non-invertible, so  $1 \in \sigma(U^*U)$  and then  $\|U\| \geq 1$ . By (50) and the second assumption of the corollary,  $\|U\| \leq 1$ , so  $\|U\| = 1$ .

By Corollary 6.8(a), there exists a finite sequence  $(b_n)_{n=1}^M$  in  $A$  such that  $(b_n)_{n=1}^M \cup (a_n)_{n=1}^{\infty}$  is a Parseval frame for  $A$ . Denote  $b = \sum_{n=1}^M b_n b_n^*$ . Since  $\mathbb{K}(A)$  and  $A$  are isomorphic as  $C^*$ -algebras, by Proposition 2.3 the sequence  $(b + \sum_{n=1}^N a_n a_n^*)_N$ , as a subsequence of an approximate unit for  $A$ , is itself an approximate unit for  $A$ .  $\square$

## REFERENCES

- [1] J. Antezana, G. Corach, M. Ruiz, D. Stojanoff, *Oblique projections and frames*, Proc. Amer. Math. Soc., 134 (2006), 1031–1037.



- [2] Lj. Arambašić, *On frames for countably generated Hilbert  $C^*$ -modules*, Proc. Amer. Math. Soc., 135 (2007). 469–478.
- [3] D. Bakić, T. Berić, *Finite extensions of Bessel sequences*, Banach J. Math. Anal., 9 (2015), no 4, 1–13.
- [4] D. Bakić, T. Berić, *On excesses of frames*, accepted for publication in Glasnik Matematički.
- [5] D. Bakić, B. Guljaš, *Extensions of Hilbert  $C^*$ -modules, I*, Houston J. Math., 30 (2004) 537–558.
- [6] D. Bakić, B. Guljaš, *On a class of module maps of Hilbert  $C^*$ -modules*, Math. Commun., 7 (2003), 177–192.
- [7] P. Casazza, O. Christensen, *Perturbations of operators and applications to frame theory*, J. Fourier Anal. Appl., 3 (1997), 543–557.
- [8] O. Christensen, *An introduction to frames and Riesz bases*, Birkhäuser, 2003.
- [9] M. Frank, D. Larson, *A module frame concept for Hilbert  $C^*$ -modules*, 207–233, Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999.
- [10] M. Frank, D. R. Larson, *Frames in Hilbert  $C^*$ -modules and  $C^*$ -algebras*, J. Operator Theory 48 (2002), no. 2, 273–314.
- [11] M. Frank, D. Larson, *Modular frames for Hilbert  $C^*$ -modules and symmetric approximation of frames*, Proc. SPIE 4119 (2000), 325–336.
- [12] D. Han, *Frame representations and Parseval duals with applications to Gabor frames*, Trans. Amer. Math. Soc., 360 (2008), 3307–3326.
- [13] D. Han, D. Larson, W. Jing, R. N. Mohapatra, *Riesz bases nad their dual modular frames in Hilbert  $C^*$ -modules*, J. Math. Anal. Appl., 343 (2008), 246–256.
- [14] D. Han, D. Larson, W. Jing, R. N. Mohapatra, *Perturbation of frames and Riesz bases in Hilbert  $C^*$ -modules*, Linear Algebra Appl., 431 (2009), no. 5–7, 746–759.
- [15] W. Jing, *Frames in Hilbert  $C^*$ -modules*, PhD thesis, University of Central Florida, 2006.
- [16] V. Kaftal, D. Larson, S. Zhang, *Operator-valued frames on  $C^*$ -modules*, Contemporary Math., 451 (2008) 363–405.
- [17] C. Lance, *Hilbert  $C^*$ -Modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge, 1995.
- [18] V. M. Manuilov and E. V. Troitsky, *Hilbert  $C^*$ -Modules*, Translations of Mathematical Monographs v. 226. American Mathematical Society, Providence, R.I., USA, 2005.
- [19] W.L. Paschke, *Inner product modules over  $B^*$ -algebras*, Trans Amer. Math. Soc., 182 (1973), 443–468.
- [20] I. Raeburn, S. J. Thompson, *Countably generated Hilbert modules, the Kasparov stabilization theorem, and frames in Hilbert modules*, Proc. Amer. Math. Soc., 131 (2003). 1557–1564.
- [21] I. Raeburn, D. P. Williams, *Morita equivalence and continuous trace  $C^*$ -algebras*, Math. Surveys and Monogr. v. 60. Amer. Math. Soc., Providence, R.I., 1998.
- [22] E. V. Troitsky, *Geometry and topology of operators on Hilbert  $C^*$ -modules*. Functional Analysis, 6, (1998). Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., v. 53, Vseross. Inst. Nauchn. i Tekhn. Inform. (VINITI), Moscow, ed.: A. Ya. Khelemskii / J. Math. Sci. 98(2000), 245–290.
- [23] N.E. Wegge-Olsen,  *$K$ -Theory and  $C^*$ -Algebras - A Friendly Approach*, Oxford Univ. Press, Oxford, 1993.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA CESTA 30, 10000 ZAGREB, CROATIA.

*E-mail address:* arambas@math.hr

*E-mail address:* bakic@math.hr